

# On global anomalies in type IIB string theory

Hisham Sati \*

Department of Mathematics  
University of Pittsburgh  
Pittsburgh, PA 15260

## Abstract

We study global gravitational anomalies in type IIB string theory with nontrivial middle cohomology. This requires the study of the action of diffeomorphisms on this group. Several results and constructions, including some recent vanishing results via elliptic genera, make it possible to consider this problem. Along the way, we describe in detail the intersection pairing and the action of diffeomorphisms, and highlight the appearance of various structures, including the Rochlin invariant and its variants on the mapping torus.

arXiv:1109.4385v1 [hep-th] 20 Sep 2011

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\*e-mail: [hsati@pitt.edu](mailto:hsati@pitt.edu)

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## 1 Introduction

There are two string theories with chiral supersymmetry in ten dimensions: heterotic string theory and type IIB string theory. Due to the presence of chiral fermions, these theories might a priori suffer from anomalies, both local and global. However, the first theory is in fact anomaly free, both locally [3] and globally [54] [55]. The second theory is also free of local anomalies [3] as seen via a “miraculous cancellation formula”. Thus, it then makes sense to discuss global anomalies. The question of whether or not there are global anomalies in type IIB string theory has been investigated by Witten in Ref. [54], in the special case when the middle cohomology vanishes. These potential anomalies are gravitational since type IIB string theory has no gauge fields, and hence there are obviously no global gauge anomalies. The aim of this paper is to investigate the question in general.

**Difficulties.** Witten’s analysis indicates that type IIB string theory does not have global anomalies, but he states that his conclusion is not quite rigorous because of physical and mathematical uncertainties about how to treat antisymmetric tensor fields. This involves encoding the antisymmetric field by an operator acting on bispinors. Furthermore, if the fifth Betti number of  $X^{10}$  is not zero then this operator might have zero modes and hence might affect the result. Therefore, the paper [54] worked with the assumption that the fifth Betti number  $b_5(X^{10})$  vanishes. In the case

when  $b_5(X^{10}) \neq 0$ , it was anticipated by Witten that the contribution of the self-dual tensor to a possible global anomaly depends on the action of the diffeomorphism  $f$  on the middle cohomology group  $H^5(X^{10}; \mathbb{R})$ .

**Recent developments.** There has been several developments since the original treatment in [54] which make it timely to revisit this problem. These developments include:

1. Much better understanding of the dynamics of self-dual fields [23] [49] [6] and their partition functions [24] [56] [57] [26] [7] [8] [36]. The bosonic field appearing in the discussion of the anomaly is the 5-form antisymmetric tensor, which is self-dual.
2. More techniques for counting fermion zero modes [4] [15] [28]. Global anomalies can involve the phase of the effective action, and can be investigated by counting the number of fermion zero modes for the Dirac operator, and the number of zero modes for the signature operator. The latter has been studied in the dual context of M-theory [46] [47].
3. An understanding of the need for a quadratic refinement of bilinear forms associated with the self-dual field [56] [57]. Global anomaly considerations will have to take such refinements into account. This will be central in our description of the structures associated with the anomaly.
4. A better understanding of the geometry of diffeomorphisms [34] [31] [18]. This includes the description of the holonomy of the line bundle associated to the signature via the Rochlin invariant and variations thereof.
5. A better understanding of structures related to the families index theorem using generalized cohomology [16], and corresponding vanishing theorems, using elliptic genera [21]. The latter will be a major point in our description; it will allow us to deduce the triviality of the holonomy of the anomaly line bundle.

In addition to the above relatively recent works, we make use of classic results in topology not widely known in the physics literature; this includes [29] [13].

**What we do.** We provide the context which brings into light the relevance and the applicability of the above works, and apply the techniques in a suggestive way that leads us naturally to arrive at the desired conclusions on global anomalies of type IIB string theory in the case when  $b_5(X^{10}) \neq 0$ . In particular, we study the anomaly line bundles, their holonomy, the effect of diffeomorphisms on the middle cohomology in ten dimensions as well as on (almost) middle cohomology in eleven and twelve dimensions, via the mapping torus and its bounding space. We view the main point as a culmination of the above works. Along the way, we clarify the physical role of the various geometric, topological and algebraic structures involved. Thus the paper takes on an expository style throughout, and in certain sections is a survey.

**Outline of the paper.** We start in section 2.1 by reviewing the basic setting in type IIB string theory; this includes the field content, the self-dual fields and their local anomalies, and the analysis of the global anomalies in the special case of vanishing middle cohomology. In section 2.2 we outline the construction of the line bundles associated with the three relevant operators: the Dirac, Rarita-Schwinger, and signature operators, using Atiyah's formulation of the latter. This then leads to a description of their holonomy in section 2.3 in the context of Bismut and Freed. Having set up the

holonomy in terms of eta invariants, we study the variation over the parameter space in section 2.4, thereby demonstrating cancellation via elliptic genera. Having spelled out the main ingredient in the anomaly cancellation, we go back and study details and tie some ends, starting in section 3, where we include the middle cohomology and study the resulting intersection forms in ten and twelve dimensions (with boundary) in section 3.1 and section 3.2, respectively. The description of the action of diffeomorphisms on middle cohomology is better done using the dual homology instead, which we explain in section 3.3. Then, in section 3.4, we bring in quadratic forms and their refinements, which are essential for studying the self-dual field. We distinguish quadratic forms appearing over  $\mathbb{Z}$ ,  $\mathbb{Z}_2$  and  $\mathbb{Q}/\mathbb{Z}$  and in ten, eleven, and twelve dimensions in section 3.5, where we also describe the connection to the Arf invariant. This leads to the study of characteristic vectors, in section 3.6, and Wu classes, in section 3.7, as they also appear in the partition function, which we use for insight. Having set up algebraic, geometric and topological tools, we apply them to the study of diffeomorphisms in section 4. We consider diffeomorphisms preserving the Spin structure and quadratic forms in section 4.1 and section 4.2, respectively. Finally in section 4.4 we describe the relation to the Rochlin invariant of the mapping torus, and in section 4.5 to the Neumann, Fischer-Kreck, and Ochanine invariants.

Many of the constructions in this paper carry over to the M5-brane, for which similar results hold. We plan to spell out the details elsewhere.

*Note added.* After we finished writing this paper, a preprint appeared [37] in which the author announces a forthcoming work on the same problem. The two approaches seem to be different, and we hope that they will each enrich the knowledge in this area. A possible connection is that a good part of our (more formal) discussion can be recast in terms of theta functions and theta multipliers, via the topological interpretation of these in [34] [31].

## 2 Global anomaly cancellation

In this section we start by reviewing the physical setting, then we set up the line bundles needed to study the global anomaly, and then we study the cancellation of that anomaly.

### 2.1 Review of the setting in type IIB string theory

We recall some of the basic aspects of type IIB string theory that we will need for the rest of the paper and which will pave the way for the discussion of global anomalies. We take type IIB string theory on a 10-dimensional Spin manifold  $X^{10}$  with metric  $g$ , tangent bundle  $TX$ , and Spin bundle  $S(X)$ .

Type IIB supergravity is the classical low energy limit of type IIB string theory. There is no manifestly Lorentz-invariant action for this theory [33], but one can write down the equations of motion [48][27], and the symmetries and transformation rules [50].

A key property of a self-dual theory, like the type IIB theory, is that there is no single preferred action, but rather there is a family of actions parametrized by a Lagrangian decomposition of the space of fields. In type IIB string theory there is no canonical choice of such Lagrangian decomposition for general spacetimes, and that is why writing an action is difficult. However, in the case of product spacetimes and at low energy, corresponding actions can be written [8].

**Field content.** The field content of type IIB supergravity is:

1. *Bosonic:* metric  $g$ , two scalars  $\phi$  and  $\chi$ , a complex 3-form field strength  $G_3$  and a real self-dual 5-form field strength  $F_5$ . Within this set, the latter field will be the main focus of this paper.
2. *Fermionic:* two gravitini  $\psi^i$  ( $i = 1, 2$ ) of the same chirality, i.e. sections of  $S(X)^\pm \otimes (TX - 2\mathcal{O})$  (with the same choice of sign), and two dilatini of the opposite chirality,  $\lambda^i \in \Gamma[S(X)^\mp]$ . Here  $\mathcal{O}$  denotes a trivial line bundle.

**Self-dual fields.** In  $10 = 4 \cdot 2 + 2$  dimensions, from a pair of spinors of the same chirality one can always construct the components of a 5-form  $F_5$  by sandwiching five (different)  $\gamma$ -matrices between the two spinors (see e.g. [2]). There are two cases to consider, according to the signature of the 10-dimensional metric:

1. *Lorentzian with metric  $g^L$ :*  $F_5^L$  is self-dual if  $F_{\mu_1 \dots \mu_5}^L = \frac{1}{5!} \epsilon_{\mu_1 \dots \mu_{10}} F_L^{\mu_6 \dots \mu_{10}}$  with  $\epsilon_{01 \dots 9} = +\sqrt{|g^L|}$  and is obtained from two spinors  $\psi_I$  ( $I = 1, 2$ ) satisfying  $\gamma_M \psi_I = +\psi_I$ , where  $\gamma_M = \gamma_M^0 \dots \gamma_M^9$  is the chirality matrix in Minkowski space.
2. *Riemannian with metric  $g^R$ :*  $F_5^R$  is called self-dual if  $F_{j_1 \dots j_5}^R = \frac{i}{5!} \epsilon_{j_1 \dots j_{10}}^R F_R^{j_6 \dots j_{10}}$  with  $\epsilon_R^{1 \dots 10} = 1/\sqrt{g^R}$  and is obtained from two spinors  $\chi_I$  ( $I = 1, 2$ ) satisfying  $\gamma_E \chi_I = +\chi_I$ , where  $\gamma_E = i\gamma_E^1 \dots \gamma_E^{10}$  is the chirality matrix in Euclidean space.

With the careful conventions in Ref. [9],  $\gamma_E = -\gamma_M$  upon analytic continuation, and so what is self-dual in one signature is anti-self-dual in the other. We will be mostly focusing on the Riemannian case for the geometric and topological considerations we have in mind.

**Local anomalies with self-dual fields.** In addition to arising from spinors, anomalies can result from a self-dual or anti-self-dual 5-form  $F_5$  in ten dimensions. Since  $F_5$  can be constructed from a pair of positive chirality spinors, the contribution to the anomaly is given by the  $\hat{A}$ -genus multiplied by  $\text{tr exp}(iR)$ , where  $R$  is the curvature of the metric  $g$ . There are two factors of  $\frac{1}{2}$ , one coming from chiral projection of the spinor and another due to the fact that  $F_5$  is real. Overall, the index density is the degree twelve form

$$I^A(R) = \frac{1}{4} \left[ \hat{A}(Z) \text{tr exp}(iR) \right] = \frac{1}{4} [L(Z)]_{(12)} , \quad (2.1)$$

where  $L(Z)$  is the Hirzebruch  $L$ -polynomial. The index of a negative chirality (anti-self-dual) field is minus that of the corresponding positive chirality (self-dual) field. Therefore, the anomaly polynomial corresponding to (anti-)self-dual form field is  $I^A = \left[ -\frac{1}{2} \frac{1}{4} L(Z) \right]_{(12)}$ . Then 10-dimensional type IIB supergravity with a self-dual 5-form field, a pair of chiral spin  $\frac{3}{2}$  Majorana-Weyl gravitinos, and a pair of anti-chiral Majorana-Weyl spin  $\frac{1}{2}$  fermions, leads to the total anomaly polynomial

$$I(R) = I^A(R) - I^{\frac{1}{2}}(R) + I^{\frac{3}{2}}(R) . \quad (2.2)$$

Here  $I^{\frac{1}{2}}(R)$  is the  $\hat{A}$ -genus and  $I^{\frac{3}{2}}(R)$  is the twisted  $\hat{A}$ -genus corresponding to the Rarita-Schwinger fields. The relative minus sign is due to the spinors being of opposite chirality. Note that  $I(R) = 0$  when all the terms are added, demonstrating that type IIB supergravity indeed has no local anomalies [3].

**Global gravitational anomalies for  $b_5(X) = 0$ .** Gravitational anomalies require working with the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  of the 10-manifold  $X^{10}$  corresponding to a diffeomorphism  $f : X^{10} \rightarrow X^{10}$ , and then lifting to a bounding 12-manifold  $Z^{12}$  with  $Y^{11} = \partial Z^{12}$ . Therefore, the study of anomalies in this case requires the use of the index theorem for manifolds with boundary, i.e. of Atiyah-Patodi-Singer (APS) type [5] and hence involves eta invariants  $\eta_D$ ,  $\eta_R$ , and  $\eta_S$  of the Dirac, the Rarita-Schwinger, and the signature operators, respectively. For a theory with  $N_D$ ,  $N_R$  and  $N_S$  chiral Dirac, Rarita-Schwinger, and self-dual tensor fields, the change in the effective action under a diffeomorphism is [54]

$$\begin{aligned} \Delta I &= \frac{\pi i}{2} \left( N_D \eta_D + N_R (\eta_R - \eta_D) - \frac{1}{2} N_S \eta_S \right) \\ &= 2\pi i \left( \frac{1}{2} N_D \text{index}(D) + \frac{1}{2} N_R (\text{index}(R) - 2 \text{index}(D)) - \frac{1}{8} N_S \sigma \right) \\ &\quad - 2\pi i \int_Z \left( \frac{1}{2} N_D \hat{A}(R) + \frac{1}{2} N_R (K(R) - 2\hat{A}(R)) - \frac{1}{8} N_S L(R) \right) \mod 2\pi i, \end{aligned} \quad (2.3)$$

where  $K(R) = I^{\frac{3}{2}}(R)$  is the Rarita-Schwinger index,  $\sigma$  is the Hirzebruch signature, and  $\eta$  is the APS defect for each of the indicated operators (cf. Section 3.2). As indicated above, for type IIB string theory the values are  $N_D = -1$ ,  $N_R = 2$ , and  $N_S = 1$ , so that

$$\Delta I = -2\pi i \frac{\sigma(Z^{12})}{8} \mod 2\pi i. \quad (2.4)$$

The quantity  $\Delta I$  is a topological invariant, since mod 16 the signature  $\sigma(Z^{12})$  depends only on the topology of  $\partial Z^{12} = (X^{10} \times S^1)_f$ . Therefore, if  $\sigma(Z^{12})$  is divisible by 8 then the effective action is invariant and hence there are no global anomalies in this case [54]. As recalled in the introduction, the above analysis is done for the case when  $b_5(X^{10}) = 0$ . What we do in the rest of the paper is extend to the case when there is nontrivial middle cohomology and then investigate the corresponding effect of the relevant diffeomorphisms.

## 2.2 Line bundles on parameter space

In this section we describe the line bundles on the parameter space which capture the contribution to the global anomaly of each of our three operators. We will consider structures related to the situation depicted in this diagram

$$\begin{array}{ccccccc} X^{10} & \longrightarrow & Y^{11} & \hookrightarrow & Z^{12} & & \mathcal{N} \longleftarrow X^{10} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & S^1 & \hookrightarrow & \Sigma & \longrightarrow & \mathcal{B} \end{array} \quad (2.5)$$

where

- $\Sigma$  is a Riemann surface with boundary  $\partial \Sigma = S^1$ .
- $Y^{11} = (X^{10} \times S^1)_f$  is the mapping torus corresponding to a diffeomorphism  $f : X^{10} \rightarrow X^{10}$ , which has the structure of a bundle over  $S^1$  with fiber the 10-manifold  $X^{10}$ .
- $\mathcal{B}$  is the parameter space which will be the product of the intermediate Jacobian and the space of metrics modulo appropriate diffeomorphisms.

- $\mathcal{N}$  will be the total space of  $\mathcal{B}$  with fiber  $X^{10}$ .

We will consider the above bundles with rescaled metrics, generically  $g_{\text{tot}} = g_{\text{fiber}} \oplus \frac{1}{\varepsilon^2} g_{\text{base}}$  and take the limit  $\varepsilon \rightarrow 0$ .

A summary of the line bundles is provided by the following:

**1. The line bundle corresponding to the Dirac and Rarita-Schwinger operators.** These are line bundles over the space of metrics modulo diffeomorphisms. The index of the Dirac operator in dimensions of the form  $8k + 4$ , corresponding to anomalies in dimensions  $8k + 2$ , is even since the spinors are quaternionic. This implies that the first Chern class of the corresponding determinant line bundle will be even [19].

**2. The line bundle corresponding to the signature operator.** This is a line bundle over the space of the fields of middle degree, that is the intermediate Jacobian  $\text{Jac}$ . With  $S^\pm$  the positive and negative chirality spinor bundles, the spaces  $S^+ \otimes S^+$  and  $S^- \otimes S^+$  are isomorphic to the spaces of even, respectively, odd self-dual forms. Therefore, a chiral Dirac operator coupled to the positive chirality Spin bundle  $S^+$  can be viewed as a signature operator.

**3. The combined line bundles.** We will consider the tensor product of the three line bundles above. However, two of them live on one space and the third lives in another. One thought would be to take the product of the two spaces, the space of metrics modulo appropriate diffeomorphisms and the space of the fields of middle degree, and declare this as the general base space of the line bundles. However, the restriction of the Hodge  $*$  operator to middle degree cohomology provides a map from  $\mathcal{M}/\mathcal{D}$  to  $\text{Jac}$ , which we can use to pull back the line bundle over the latter.

We now consider the line bundles in more detail. Since the ‘usual’ Dirac operator is familiar, we will focus on the signature operator (which itself can be viewed as a Dirac operator), in the setting of diagram (2.5).

**The line bundle corresponding to the signature operator.** Let  $\pi : X^{10} \rightarrow \mathcal{N} \rightarrow \mathcal{B}$  be a smooth fibration with fiber at a point  $x \in \mathcal{B}$  a 10-manifold  $X_x^{10}$  which is equipped with a metric and a compatible Spin structure. The Spin structure varies smoothly over the parameter space  $\mathcal{B}$  so that the structure group of the fibration  $\pi$  is a subgroup of the Spin diffeomorphism group. Then there is a principal  $\text{Spin}(10)$  bundle  $P(X_x^{10})$  over the fibers. With  $S^\pm$  the positive and negative chirality half-spinor representations of  $\text{Spin}(10)$ , we form the vector bundles  $E_x^\pm = P(X_x^{10}) \otimes S^\pm$  and the corresponding Dirac operator  $D_x^A : L^2(E_x^\pm) \rightarrow L^2(E_x^\mp)$  on the Hilbert spaces of sections. As the parameter  $x$  varies in  $\mathcal{B}$ , the Hilbert spaces of sections  $L^2(E_x^\pm)$  form Hilbert bundles  $L^2(E^\pm)$  and the operators  $D_x^A$  form a continuous family of operators  $D : L^2(E^\pm) \rightarrow L^2(E^\mp)$  on these Hilbert bundles.

There is a well-defined complex line bundle  $\det D^A$  over  $\mathcal{B}$ . The fiber  $(\det D)_x^A$  over a point  $x \in \mathcal{B}$  is isomorphic to the space  $(\Lambda^{\max} \ker D_x^A)^* \otimes (\Lambda^{\max} \text{coker } D_x^A)$ . There is a connection  $\nabla$  on the line bundle  $\det D^A$  over  $\mathcal{B}$  whose holonomy<sup>2</sup> around an immersed circle  $\gamma : S^1 \rightarrow \mathcal{B}$  in the base manifold can be described as follows: Pulling back by  $\gamma$  there is an 11-dimensional manifold which is diffeomorphic to the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  with a diffeomorphism  $f$  specified by  $\gamma$ . Choosing an arbitrary metric  $g_{S^1}$  on  $S^1$ , and using the projection  $\text{pr} : TY^{11} \rightarrow T_Y Y^{11}$  to the tangent bundle along the fibers, we obtain a Riemannian structure on  $Y^{11}$ . Since the structure

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<sup>2</sup>We will describe the holonomy more fully in the next section.

group of the fibration  $\pi$  is a subgroup of the Spin diffeomorphism group, it follows that  $f$  is covered by a canonical Spin diffeomorphism<sup>3</sup> and the mapping torus has a natural Spin structure. From this Spin structure on  $Y^{11}$  we obtain a Spin bundle over  $Y^{11}$  with structure group  $\text{Spin}(11)$  and a corresponding Dirac operator on the space of smooth sections of this bundle.

**The signature of the extension of the mapping torus.** Consider the fibration  $X^{10} \rightarrow Z^{12} \rightarrow \Sigma$ , where  $\Sigma$  is a Riemann surface with boundary. Assume that the total space  $Z^{12}$  is oriented; this is equivalent to assuming that the fundamental group  $\pi_1(\Sigma)$  acts trivially on  $H^{10}(X^{10})$ . Then the signature  $\sigma(Z^{12})$  on the middle cohomology  $H^6(Z^{12})$  is defined. Assuming appropriate metrics, we have an APS problem and the signature is given by the APS index theorem. The extension of the bundle structure from the mapping torus  $Y^{11}$  to its bounding space  $Z^{12}$  involves taking into account cobordism of diffeomorphisms, discussed in section 4.5.

**The signature of the 10-manifold  $X^{10}$ .** Consider the signature operator  $\mathcal{S}_X$  of the 10-manifold  $X^{10}$  defined as  $\mathcal{S}_X = d + d^\dagger : \Omega^+(X^{10}) \rightarrow \Omega^-(X^{10})$ , where  $\Omega^\pm$  are the  $\pm 1$ -eigenspaces of the involution  $\omega_p \rightarrow i^{p(p-1)+1} * \omega_p$  on a  $p$ -form  $\omega_p$ . We see that for  $p = 5$ , the involution is  $\omega_5 \mapsto i * \omega_5$ . Let  $H^+$  and  $H^-$  denote the solution spaces of  $\mathcal{S}_X u = 0$  and  $\mathcal{S}_X^\dagger v = 0$ , respectively, i.e. the spaces of harmonic forms in  $\Omega^+(X^{10})$  and in  $\Omega^-(X^{10})$ . Now if we vary  $X^{10}$  over the fibers of  $Z^{12} \rightarrow \Sigma$  we get a family  $D_x$  of Dirac operators and corresponding spaces  $H_x^+$  and  $H_x^-$  of harmonic forms which define vector bundles  $H^+$  and  $H^-$  over  $\Sigma$ . The Quillen line bundle  $\mathcal{L}$  is the bundle  $\det H^- \otimes (\det H^+)^{-1}$  over  $\Sigma$  endowed with a natural unitary connection.

**Zero modes of the signature operator.** As explained in [4], one advantage of the signature operator over the generic Dirac operator is the ability of the former to control the integer ambiguity left by the Bismut-Freed formulation. This is because the zero eigenvalue of the Dirac operator cannot be controlled in general, while for the signature operator the identification of harmonic forms with cohomology via Hodge theory fixes the integer ambiguity. The 0-eigenvalues of the signature operator, given by the harmonic bundles  $H^\pm$ , can be incorporated as follows (see [4]). Let  $\mathcal{S}'_X$  be the restriction of the signature operator  $\mathcal{S}_X$  to the orthogonal complement of the harmonic spaces  $H^\pm$ . Then, via Quillen's formalism,  $\det \mathcal{S}'_X$  is a nowhere zero section of a line bundle  $\mathcal{L}'$  with a unitary connection over  $\Sigma$ . The harmonic bundles  $H^\pm$  have natural metrics and connections induced via orthogonal projection from the Hilbert space bundles of all forms. Then  $\mathcal{H} = \det H^- \otimes (\det H^+)^{-1}$  is a line bundle with unitary connection. The two line bundles are then related as  $\mathcal{L} = \mathcal{L}' \otimes \mathcal{H}$  with the induced unitary connection.

### 2.3 Holonomy of line bundles on the parameter space

All three operators that we have, namely the Dirac operator, the Rarita-Schwinger operator, and the signature operator are of Dirac-type, that is are examples of generalized Dirac operators. In this section we consider the holonomy of the line bundles associated with these operators on the parameter space, using the general formulation of Bismut and Freed [10] [11].

**Holonomy of the line bundle.** In order for the eta invariants to be independent of the metric on  $S^1$ , we rescale the metric on the circle  $g_{S^1}$  as  $\frac{1}{\epsilon} g_{S^1}$  and take the adiabatic limit, given by  $\epsilon \rightarrow 0$ .

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<sup>3</sup>See Section 4.1 for details.



Corresponding to the rescaled metric we have a Dirac operator  $D_\epsilon^A$  on the mapping torus  $Y^{11}$  and an eta invariant  $\eta(D_\epsilon^A)$ . We form the reduced eta invariant as  $\bar{\eta}(D_\epsilon^A) = \frac{1}{2}(\eta(D_\epsilon^A) + \dim \ker D_\epsilon^A) = \frac{1}{2}(\eta(D_\epsilon^A) + h(D_\epsilon^A))$ , where  $h$  is the number of zero modes. Then the Bismut-Freed theorem [10] [11] says that the holonomy around the loop  $\gamma$  of the connection  $\nabla$  on the determinant line bundle is

$$\text{hol}(\gamma; \det D^A, \nabla) = \lim_{\epsilon \rightarrow 0} e^{-2\pi i \bar{\eta}(D_\epsilon^A)} . \quad (2.6)$$

The above has been for the signature operator (viewed as a generalized Dirac operator). There are similar results with obvious changes for the Dirac operator and the Rarita-Schwinger operator; for the latter we have to replace the Spin bundle by the tangent bundle. Let us denote the resulting three line bundles with connections by  $\mathcal{L}_A$ ,  $\mathcal{L}_{RS}$ , and  $\mathcal{L}_{\text{Dir}}$ , corresponding to the signature, the Rarita-Schwinger operator, and the Dirac operator, respectively. The holonomy of each of the connections on the line bundles corresponding to the three operators will have expressions of the form (2.6). The holonomy of the tensor product line bundle <sup>4</sup>

$$\mathcal{L}_{\text{tot}} := \mathcal{L}_A \otimes \mathcal{L}_{RS}^{-1} \otimes \mathcal{L}_{\text{Dir}}^4 \quad (2.7)$$

with tensor product connection  $\nabla^{\text{tot}}$  will take the form

$$\text{hol}(\gamma; \mathcal{L}_{\text{tot}}, \nabla^{\text{tot}}) = \lim_{\epsilon \rightarrow 0} \exp \left\{ -2\pi i \left[ \bar{\eta}(D_\epsilon^A) - 8\bar{\eta}(D_\epsilon^{RS}) + 32\bar{\eta}(D_\epsilon) \right] \right\} . \quad (2.8)$$

**Line bundles over  $S^1$  vs. over  $\Sigma$ .** The first Chern form of the Quillen line bundle  $\mathcal{L}$  is [10] [11]

$$c_1(\mathcal{L}) = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{X^{10}} L_{12} \quad (2.9)$$

where the factor of  $\frac{1}{2}$  arises because we are dealing with the  $L$ -polynomial rather than the  $\hat{A}$ -genus. When  $\Sigma$  is the disk  $\mathbb{D}^2$ , the holonomy of  $\mathcal{L}$  around the bounding circle of  $\Sigma$  is just  $\exp(-\pi i \eta^0(Y^{11}))$ , where  $\eta^0 = \lim_{\epsilon \rightarrow 0} \eta^\epsilon$  is the adiabatic limit of the eta invariant. For global anomalies we consider  $\Sigma$ 's that are topologically nontrivial. The extension from bundles over  $S^1$  to bundles over  $\Sigma$  will be discussed in section 4.5.

**Relative Chern class of the holonomy line bundle.** As  $\mathcal{L}'$  (from the end of last section) is trivialized by  $\det S'_X$ , we have an isomorphism  $\mathcal{L} \cong \mathcal{H}$  as a bundle but the isomorphism does *not* preserve the metric or connection. Using expression (2.9), the APS index formula can be written as

$$\sigma(Z^{12}) = -2 \int_{\Sigma} c_1(\mathcal{L}) - \eta^0(Y^{11}) . \quad (2.10)$$

Since, via [10],  $-\pi i \eta^0(Y^{11})$  is distinguished choice for the logarithm of the holonomy of  $\mathcal{L}$  around  $S^1 = \partial \Sigma$ , we get a *relative* Chern class  $c_1(\mathcal{L}, \eta)$ , where as explained in [4] the notation highlights that this Chern class is obtained from the eta invariant. Then (2.10) becomes

$$\sigma(Z^{12}) = -2c_1(\mathcal{L}, \eta) . \quad (2.11)$$

---

<sup>4</sup>Note that the signature is divisible by 8 (cf. section 3.6), which is ‘built into’  $\mathcal{L}_A$ . See the remarks at the end of this section for more on this. As cited at the end of the introduction, the new work [37] constructs line bundles explicitly from physical data.

This can be interpreted as signature of a local coefficient system over  $\Sigma$ . The fibration  $Z^{12} \rightarrow \Sigma$  gives a local coefficient system corresponding to the representation of the fundamental group  $\pi_1(\Sigma)$  on the cohomology of the fiber  $H^*(X^{10})$ . The middle cohomology  $H^5(X^{10})$  gives a flat bundle with an antisymmetric form. This form can be changed to a Hermitian form by complexifying coefficients and multiplying by  $i$ . This Hermitian form has type  $(\frac{1}{2}b_5, \frac{1}{2}b_5)$ , where  $b_5 = \dim H^5(X^{10})$  is the fifth Betti number of the fiber. From [4], multiplicativity of the signature for fiber bundles gives that the signature of  $\Sigma$  with coefficients in this flat bundle is equal to the signature of the total space

$$\sigma(\Sigma, H^5(X^{10})) = \sigma(Z^{12}). \quad (2.12)$$

The contribution to  $H^\pm$  from  $H^j(X^{10})$  and  $H^{5-j}(X^{10})$  for  $j \neq 5$  cancel. That is, there is no contribution from the Ramond-Ramond fields other than the self-dual 5-form.

As a warm-up for the general discussion in Section 2.4 below, we illustrate some of the points on the variation of the above holonomy in the case of change of Spin structure.

**Different Spin structures.** Suppose that our 10-manifold  $X^{10}$  has more than one Spin structure (see [45] for an extensive discussion of the effect of multiple Spin structures in the related context of M-theory). Suppose  $\pi$  is a fibration of 10-manifolds  $X^{10}$ , with two preferred Spin structures  $\omega_1$  and  $\omega_2$ . Corresponding to these two Spin structures there are families of Dirac operators  $D_{\omega_1}$  and  $D_{\omega_2}$ , and corresponding determinant line bundles  $\det D_{\omega_1}$  and  $\det D_{\omega_2}$ . From the curvature formula of Bismut-Freed, these two complex line bundles have the same curvature 2-form. Hence, the contribution of their difference to the local anomaly is zero, and hence the local anomaly is not sensitive to the change of Spin structure. Stated more precisely, the complex line bundle  $\mathcal{L}_\omega = \det D_{\omega_1} / \det D_{\omega_2} = \det D_{\omega_1} \otimes (\det D_{\omega_2})^*$  is flat. However, for the global anomaly we need to investigate the holonomies of this flat line bundle. Let  $\omega'_1$  and  $\omega'_2$  be the Spin structures on the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  induced by the Spin structures  $\omega_1$  and  $\omega_2$  on  $X^{10}$ . The corresponding diffeomorphism  $f$  is specified by the loop  $\gamma$ . Then, from [31], we have that the holonomy of the the quotient line bundle  $\mathcal{L}_\omega$  is given by

$$\text{hol}(\gamma; \mathcal{L}_\omega, \nabla_\omega) = \lim_{\epsilon \rightarrow 0} \exp \left\{ -2\pi i \left[ \bar{\eta}(D_{\omega'_1}^\epsilon) - \bar{\eta}(D_{\omega'_2}^\epsilon) \right] \right\}. \quad (2.13)$$

This formula holds for all three operators, namely the Dirac operator, the Rarita-Schwinger operator, and the signature operator. Hence, the total variation with respect to the change of Spin structure is given essentially by the product of three appropriate copies of either side of expression (2.13). We need to consider the change of  $\mathcal{L}_{\text{tot}}$  under the variation of Spin structure. This will then be (we use  $\text{hol}$  to denote  $\text{hol}(\gamma; \mathcal{L}_{\text{tot}}, \nabla_\omega^{\text{tot}})$ )

$$\begin{aligned} \text{hol} &= \lim_{\epsilon \rightarrow 0} \exp \left\{ -2\pi i \left[ (\bar{\eta}(D_{\epsilon, \omega_1}^A) - \bar{\eta}(D_{\epsilon, \omega_2}^A) - 8(\bar{\eta}(D_{\epsilon, \omega_1}^{RS}) - \bar{\eta}(D_{\epsilon, \omega_2}^{RS})) + 32(\bar{\eta}(D_{\epsilon, \omega_1}) - \bar{\eta}(D_{\epsilon, \omega_2}))) \right] \right\} \\ &= \lim_{\epsilon \rightarrow 0} \exp \left\{ -2\pi i \left[ (\bar{\eta}(D_{\epsilon, \omega_1}^A) - 8\bar{\eta}(D_{\epsilon, \omega_1}^{RS}) + 32\bar{\eta}(D_{\epsilon, \omega_1})) - (\bar{\eta}(D_{\epsilon, \omega_2}^A) - 8\bar{\eta}(D_{\epsilon, \omega_2}^{RS}) + 32\bar{\eta}(D_{\epsilon, \omega_2})) \right] \right\} \\ &= \lim_{\epsilon \rightarrow 0} \exp \left\{ -2\pi i [\bar{\eta}_{\text{tot}}^{\omega_1} - \bar{\eta}_{\text{tot}}^{\omega_2}] \right\}, \end{aligned}$$

where we have defined the ‘total reduced eta invariant’  $\bar{\eta}_{\text{tot}}^{\omega_i} := \bar{\eta}(D_{\epsilon, \omega_i}^A) - 8\bar{\eta}(D_{\epsilon, \omega_i}^{RS}) + 32\bar{\eta}(D_{\epsilon, \omega_i})$  for  $i = 1, 2$ , corresponding to the two Spin structures. We will consider such a combination again towards the end of next section. We will demonstrate, using the results of Ref. [21], that this combination of eta invariants is constant under change of metric modulo appropriate diffeomorphisms.

## 2.4 The global anomaly cancellation

The local anomaly involves characteristic classes and characteristic forms. The global anomaly will also involve curvatures of line bundles via the holonomy (see [20] for an excellent general discussion). Modular properties of characteristic forms following from elliptic genera are powerful in giving relations among such forms. Along these lines, we will use the recent vanishing results of Ref. [21] throughout this section. That elliptic genera appear in a fundamental way in type IIB string theory is remarkable as it shows that they might have a role to play in type IIB, which is analogous to the role the Witten genus plays in anomaly cancellation in heterotic string theory [32] and in understanding topological aspects of M-theory [43] [44].

Consider the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  corresponding to a diffeomorphism  $f : X^{10} \rightarrow X^{10}$  on the type IIB spacetime  $X^{10}$ , a Spin 10-manifold. Take this mapping torus to be the fiber in the smooth fiber bundle  $Y^{11} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{\text{met}}/\mathcal{D}$  over  $\mathcal{M}_{\text{met}}/\mathcal{D}$ , the quotient of the space of Riemannian metrics  $\mathcal{M}_{\text{met}}$  on  $X^{10}$  by an appropriate diffeomorphism group  $\mathcal{D}$ . We will be interested in  $\mathcal{D}$  being the group of diffeomorphisms preserving the Spin structure on  $X^{10}$  and/or preserving the quadratic refinement corresponding to the self-dual 5-form field. We will discuss such points extensively in section 4.

Let  $TY^{11}$  with metric  $g_Y$  be the tangent bundle of the mapping torus viewed as the vertical tangent bundle of the fiber bundle  $\mathcal{M}$ . The total tangent bundle to  $\mathcal{M}$  splits orthogonally as  $T\mathcal{M} = T_H\mathcal{M} \oplus TY^{11}$ , where  $T_H\mathcal{M}$  is the smooth horizontal subbundle. A metric  $g_D$  on  $T(\mathcal{M}_{\text{met}}/\mathcal{D})$  can be lifted to a metric on  $T\mathcal{M}$  which is the sum  $g_D \oplus g_Y$ .

We need connections on the various spaces. First we start with the Levi-Civita connection  $\nabla^L$  on the tangent bundle  $T\mathcal{M}$  of the total space, and then we form the metric-preserving connection  $\nabla^Y$  on the vertical tangent bundle  $TY^{11}$  defined by the relation  $\nabla_U^Y V = P_Y \nabla_U^L V$ , for  $U \subset T\mathcal{M}$ ,  $V \subset TY^{11}$ . Here  $P_Y$  is the orthogonal projection from the total tangent bundle  $T\mathcal{M}$  to the vertical tangent bundle  $TY^{11}$ . In order to consider characteristic forms and characteristic classes we form the curvature  $R^Y = (\nabla^Y)^2$  of the connection  $\nabla^Y$ .

**The family signature operator.** Let  $\{e_1, e_2, \dots, e_{11}\}$  be an oriented orthogonal basis of  $TY^{11}$ . We can form the exterior bundle  $\Lambda TY^{11}$  and consider differential forms on the mapping torus  $Y^{11}$ . Let  $d^Y$  denote the exterior derivative along the fibers. Denote by  $c$  the Clifford action on the complexified exterior algebra bundle  $\Lambda_{\mathbb{C}}(T^*Y^{11})$  of the cotangent bundle  $T^*Y^{11}$  of the fiber. On an element  $e$ , this is given by  $c(e) = e^* - i_e$ , where  $e^*$  is the dual element in  $T^*Y^{11}$  via  $g_Y$  and  $i_e$  is contraction with the vector  $e$ . The chirality operator  $\Gamma = -c(e_1) \cdots c(e_{11})$  is a self-adjoint element satisfying  $\Gamma^2 = \text{Id}$ . Define the family odd signature operator  $\mathcal{S}^Y$

$$\mathcal{S}^Y = \Gamma d^Y + d^Y \Gamma : C^\infty(\mathcal{M}, \Lambda_{\mathbb{C}}^{\text{ev}}(T^*Y^{11})) \rightarrow C^\infty(\mathcal{M}, \Lambda_{\mathbb{C}}^{\text{ev}}(T^*Y^{11})). \quad (2.14)$$

For each point in the base space  $x \in \mathcal{M}_{\text{met}}/\mathcal{D}$  corresponding to an equivalence class of metrics, the restriction to the fiber over this point

$$\mathcal{S}_x^Y : C^\infty(Y_x^{11}, \Lambda_{\mathbb{C}}^{\text{ev}}(T^*Y^{11})|_x) \rightarrow C^\infty(Y_x^{11}, \Lambda_{\mathbb{C}}^{\text{ev}}(T^*Y^{11})|_x) \quad (2.15)$$

is the odd signature operator for the fiber  $Y_x^{11}$  (cf. [5]).

**The family (twisted) Dirac operator.** Assume that  $TY^{11}$  is Spin and form the Spin bundle  $S(Y^{11})$ . Consider the twisting of the Spin bundle by the complexified tangent bundle  $V = T_{\mathbb{C}}Y^{11}$ , that is  $S(Y^{11}) \otimes T_{\mathbb{C}}Y^{11}$ . On this twisted bundle we have a connection  $\nabla^V$  and a twisted Dirac operator  $D^Y \otimes V = \sum_{i=1}^{11} e_i \nabla_{e_i}^V$ . As in section 2.3 above, for  $x \in \mathcal{M}_{\text{met}}/\mathcal{D}$ , let  $\eta_x(D^Y \otimes V)$  be the eta invariant corresponding to the twisted Dirac operator and consider the reduced eta invariant  $\overline{\eta}_x(D^Y \otimes V) = \frac{1}{2} (\eta_x(D^Y \otimes V) + \dim \ker(D^Y \otimes V)_x)$ , as a function on  $\mathcal{M}_{\text{met}}/\mathcal{D}$ .

**Consequences of modular invariance from elliptic genera.** We will review the results of [21] and provide an interpretation. Let  $\widetilde{T_{\mathbb{C}}Y^{11}} = T_{\mathbb{C}}Y^{11} - \dim T_{\mathbb{C}}Y^{11}$  be the reduced element in K-theory of the total space  $K(\mathcal{M})$  corresponding to the complexified vertical tangent bundle. Following [21], define the  $q$ -expansion

$$\Theta_2(T_{\mathbb{C}}Y^{11}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}Y^{11}}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}Y^{11}}) \in K(\mathcal{M})[[q^{\frac{1}{2}}]] . \quad (2.16)$$

The local anomaly cancellation formula can be written as

$$\{L(TY^{11}, \nabla^Y)\}^{(12)} = 8 \sum_{r=0}^1 2^{6-6r} \left\{ \hat{A}(TY^{11}, \nabla^Y) \text{ch}(b_r(T_{\mathbb{C}}Y^{11})) \right\}^{(12)} , \quad (2.17)$$

where  $b_r(T_{\mathbb{C}}Y^{11})$  are virtual vector bundles defined by the congruence

$$\Theta_2(T_{\mathbb{C}}Y^{11}) \equiv \sum_{r=0}^1 b_r(T_{\mathbb{C}}Y^{11}) (8\delta_2)^{3-2r} \varepsilon_2^r \pmod{q \cdot K(\mathcal{M})[[q^{\frac{1}{2}}]]} . \quad (2.18)$$

Here  $\delta_2$  and  $\varepsilon_2$  are the modular forms written in terms of Jacobi theta functions with Fourier expansions in  $q^{\frac{1}{2}}$  and are given by the expressions

$$\begin{aligned} \delta_2(\tau) &= -\frac{1}{8}(\theta_1^4 + \theta_3^4) = -\frac{1}{8} - 3q^{\frac{1}{2}} - 3q + \dots , \\ \varepsilon_2(\tau) &= \frac{1}{16}\theta_1^4\theta_3^4 = q^{\frac{1}{2}} + 8q + \dots . \end{aligned}$$

**Global anomaly cancellation via the family index.** We have three family operators to consider: The Dirac operator, the twisted Dirac operator, and the odd signature operator.<sup>5</sup> Using [16], the family index of the odd signature operator on the oriented bundle  $Y^{11} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{\text{met}}/\mathcal{D}$  is trivial, that is,  $\text{ind}(\mathcal{S}^Y) = 0 \in K^1(\mathcal{M}_{\text{met}}/\mathcal{D})$ . Since the integral over the fiber  $\int_{Y^{11}} \hat{A}(TY^{11}, \nabla^Y) \text{ch}(V, \nabla^V)$  represents the odd Chern character of the index  $\text{ind}(D^Y \otimes V)$ , then the degree one class  $[\int_{Y^{11}} L(TY^{11}, \nabla^Y)]$  is zero in de Rham cohomology. The results of Bismut-Freed [10] [11] imply that

$$d\{\overline{\eta}_x(D^Y \otimes V)\} = \left\{ \int_{Y^{11}} \hat{A}(TY^{11}, \nabla^Y) \text{ch}(V, \nabla^V) \right\}^{(1)} . \quad (2.19)$$

---

<sup>5</sup>As we mentioned earlier, the (family) signature operator itself can be viewed as a twisted (family) Dirac operator.

Integrating both sides of (2.17) over the fiber  $Y^{11}$  gives

$$\left\{ \int_{Y^{11}} L(TY^1, \nabla^Y) \right\}^{(1)} - 8 \sum_{r=0}^1 2^{6-6r} \left\{ \int_{Y^{11}} \hat{A}(TY^{11}, \nabla^Y) \text{ch}(b_r(T_{\mathbb{C}}Y^{11})) \right\}^{(1)} = 0, \quad (2.20)$$

so that

$$d\{\bar{\eta}(\mathcal{S}^Y)\} - 8 \sum_{r=0}^1 2^{6-6r} d\{\bar{\eta}(D^Y \otimes b_r(T_{\mathbb{C}}Y^{11}))\} = 0. \quad (2.21)$$

Since  $\mathcal{M}_{\text{met}}/\mathcal{D}$  is connected, this implies— still applying [21]— that the combination

$$\begin{aligned} \bar{\eta}_{\text{tot}} &:= \{\bar{\eta}(\mathcal{S}^Y)\} - 8 \sum_{r=0}^1 2^{6-6r} \{\bar{\eta}(D^Y \otimes b_r(T_{\mathbb{C}}Y^{11}))\} \\ &= \bar{\eta}(\mathcal{S}^Y) - 8\bar{\eta}(D^Y \otimes T_{\mathbb{C}}Y^{11}) + 24\bar{\eta}(D^Y) \end{aligned}$$

is a constant function on the base  $\mathcal{M}_{\text{met}}/\mathcal{D}$ . Therefore, also the exponential  $\exp(2\pi i \bar{\eta}_{\text{tot}})$  is a constant function on the base. That is, the phase is invariant under the variation of the metric modulo (appropriately chosen) diffeomorphisms. We interpret this as saying that there are no global gravitational anomalies.

**Remarks. 1.** In the above formal proof, there was nothing special about the base being explicitly  $\mathcal{M}_{\text{met}}/\mathcal{D}$ . In fact, the results hold for any connected base. However, the choice we made is the one appropriate for global anomalies in type IIB string theory.

**2.** In addition, no detailed knowledge about the geometry of the base is needed. However, in order to illustrate the point, in the following sections we will include such aspects in order to describe the details of the anomaly cancellation in relation to the physical entities involved.

**3.** We have left  $\mathcal{D}$  generic for diffeomorphisms. We will be interested in diffeomorphisms which preserve the Spin structure and/or those which preserve the quadratic refinements (the two diffeomorphisms are related). Again, in order to illustrate the process physically we will describe such diffeomorphisms explicitly in section 4.

**The space of Riemannian metrics and its quotients.** The space  $\mathcal{M}_{\text{met}}$  of all Riemannian metrics  $g_X$  on  $X^{10}$  is a contractible open cone inside the space  $\Gamma(S^2 T^* X^{10})$  of symmetric rank-2 tensor fields. The group  $\text{Diff}^+(X^{10})$  of orientation-preserving diffeomorphisms acts isometrically on  $\mathcal{M}_{\text{met}}$ . This action is free on the subset  $\mathcal{M}_{\text{met}}^{\text{noniso}}(X^{10})$  of metrics which admit no nontrivial isometries. See [17] for more details. If we insist on having a smooth quotient then we should use this latter quotient for the moduli space of metrics.

**Remarks. 1.** The construction of the line bundle whose section is the partition function is more involved since it is essentially Chern-Simons theory at level  $\frac{1}{2}$  and hence requires taking delicate square roots (see [56] [57] [26] [7] [8] [36]).

**2.** For purposes of global anomalies one shows that the phase of the form  $e^{2\pi i \vartheta(x)/n}$  is constant over the moduli space of parameters,  $x$ . However, if  $e^{2\pi i \vartheta(x)}$  is constant then so will be its  $n$ th roots for

any  $n$ .<sup>6</sup>

Having spelled out the main formal argument, we now turn to some of the details involving how the global anomaly cancels in our setting, as well as illuminating details involving the physics, and highlight some interesting consequences. That is, even though the global anomaly cancellation did not care much about the details of the cancellation, it is nonetheless useful to see how the anomaly cancels. We view this as conceptually analogous to the discussion in [32] in the case of the heterotic string.

### 3 Intersection pairings in 10, 11, and 12 dimensions

We will focus on the case  $b_5(X^{10}) \neq 0$ , so that we have nontrivial cohomology  $H^5(X^{10}; \mathbb{R})$ . We will also consider extensions of this in two directions. The first is to consider the lift to the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  and to the bounding 12-manifold  $Z^{12}$  and then study the corresponding cohomology groups in these two other dimensions. The second extension is to consider integral coefficients and separate the free and the torsion parts of the corresponding cohomology groups in all three relevant dimensions. We will see that our setting will dictate preferences from the two sets of extensions.

**Identifying the intersection pairings in the relevant dimensions.** Let  $M$  be a closed oriented  $m$ -manifold. Define  $T^k(M) := T^k(M; \mathbb{Z})$  to be the torsion subgroup of the cohomology group  $H^k(M; \mathbb{Z})$ , i.e.

$$T^k(M) = H^k(M; \mathbb{Z})_{\text{tors}} = \{\alpha \in H^k(M; \mathbb{Z}) \mid r\alpha = 0 \text{ for some } r \in \mathbb{Z}\}. \quad (3.1)$$

The quotient<sup>7</sup>  $\text{Fr}^k(M) = H^k(M; \mathbb{Z})/T^k(M)$  is then a free abelian group. The pairing

$$I : H^i(M; \mathbb{Z}) \otimes H^{m-i}(M; \mathbb{Z}) \rightarrow H^m(M; \mathbb{Z}) = \mathbb{Z} \quad (3.2)$$

induces a nonsingular pairing of free groups

$$I_F : \text{Fr}^i(M) \otimes \text{Fr}^{m-i}(M) \rightarrow \mathbb{Z}. \quad (3.3)$$

There is also the nonsingular torsion pairing for  $i \neq 0$

$$L : T^i(M) \otimes T^{m+1-i}(M) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (3.4)$$

Now we would like to concentrate on the cohomology of degrees 5 and 6 and, in the closed case, on spacetime dimensions 10 and 11; we would like to consider  $X^{10}$  and its mapping torus  $Y^{11} = (X^{10} \times S^1)_f$ . In order to get an intersection form on middle cohomology of  $X^{10}$ , it is obvious that we have to look at the pairing (3.2) or at the pairing (3.3). This identifies for us the relevant pairings for  $X^{10}$ .

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<sup>6</sup>In using such an argument, some torsion information will be lost. Since  $c_1(\mathcal{L}^n) = nc_1(\mathcal{L})$ , it could happen that this is zero just because  $c_1(\mathcal{L})$  is an  $m$ -torsion class for  $m$  a divisor of  $n$ . Therefore, our arguments work best when the Chern classes of the line bundles are not torsion. Such information requires working with K-theory, which is beyond the scope of this paper.

<sup>7</sup>In general there is a short exact sequence  $0 \rightarrow T^k(M) \rightarrow H^k(M; \mathbb{Z}) \rightarrow \text{Fr}^k(M) \rightarrow 0$ .

Next, for  $Y^{11}$ , these two pairings do not give the correct degree, but instead expression (3.4) does, due to the shift of one in degree. Therefore, in eleven dimensions we consider the torsion pairing

$$L : T^6(Y^{11}) \otimes T^6(Y^{11}) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (3.5)$$

Of course we will also have the pairing on the free part, namely  $\text{Fr}^5(Y^{11}) \otimes \text{Fr}^6(Y^{11}) \rightarrow \mathbb{Z}$ .

Next we consider manifolds with boundary. Here our main case is the bounding 12-manifold  $Z^{12}$  with  $\partial Z^{12} = Y^{11}$ , the mapping torus, and the main cohomology degree is 6. The cohomology pairing  $H^6(Z^{12}; \mathbb{Z}) \otimes H^6(Z^{12}, Y^{11}; \mathbb{Z}) \rightarrow H^{12}(Z^{12}, Y^{11}; \mathbb{Z}) = \mathbb{Z}$  defines a nonsingular pairing of free abelian groups  $\text{Fr}^6(Z^{12}) \otimes \text{Fr}^6(Z^{12}, Y^{11}) \rightarrow \mathbb{Z}$ . Since we are not particularly interested in degree 7 cohomology, we will not consider a torsion pairing for  $Z^{12}$ .

We would like to consider the symmetry of the relevant pairings identified above. Useful references on bilinear and quadratic forms include [35] [52]. we will need the following notions.

**Symmetry of quadratic forms over  $\mathbb{R}$ .** For  $\epsilon = +1$  or  $-1$ , an  $\epsilon$ -symmetric form  $(V, \phi)$  is a finite-dimensional real vector space  $V$  together with a bilinear pairing  $\phi : V \times V \rightarrow \mathbb{R}$  sending  $(x, y) \mapsto \phi(x, y)$  such that  $\phi(x, y) = \epsilon \phi(y, x) \in \mathbb{R}$ . The form is called *symmetric* for  $\epsilon = +1$  and *symplectic* for  $\epsilon = -1$ . The pairing  $\phi$  can be identified with the adjoint linear map to the dual vector space  $\phi : V \rightarrow V^* = \text{Hom}(V, \mathbb{R})$  sending  $x$  to  $(y \mapsto \phi(x, y))$  such that  $\phi^* = \epsilon \phi$ . The form  $(V, \phi)$  is nonsingular if  $\phi : V \rightarrow V^*$  is an isomorphism. A *Lagrangian* of a nonsingular form  $(V, \phi)$  is a subspace  $L \subset V$  such that  $L = L^\perp$ , i.e.  $L = \{x \in V \mid \phi(x, y) = 0 \text{ for all } y \text{ in } L\}$ . The *hyperbolic  $\epsilon$ -symmetric form* is defined for any finite-dimensional real vector space  $L$  by  $\mathbb{H}_\epsilon(L) = (L \oplus L^*, \phi = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix})$ , where  $\phi : (L \oplus L^*) \times (L \oplus L^*) \rightarrow \mathbb{R}$  is given by  $((x, f), (y, g)) \mapsto g(x) + \epsilon f(y)$  with Lagrangian  $L$ . The inclusion  $L \rightarrow V$  of a Lagrangian in a nonsingular  $\epsilon$ -symmetric form  $(V, \phi)$  extends to an isomorphism  $\mathbb{H}_\epsilon(L) \xrightarrow{\cong} (V, \phi)$ .

The pairing on the middle cohomology of closed oriented  $2k$ -manifolds is symmetric for  $k$  even and antisymmetric for  $k$  odd. Therefore, in ten dimensions we will have symplectic forms corresponding to  $\epsilon = -1$ , and in twelve dimensions we will have symmetric forms corresponding to  $\epsilon = 1$ . The Lagrangian identifies the set of cohomology classes for which the intersection form is zero.

The cohomology of the pair  $(Z^{12}, Y^{11})$  gives the following diagram, which summarizes the relations between the various cohomology groups we are considering

$$\begin{array}{ccccccc}
 T^6(Z^{12}, Y^{11}) & \xrightarrow{j} & T^6(Z^{12}) & \xrightarrow{i} & T^6(Y^{11}) & \xrightarrow{\delta} & T^7(Z^{12}, Y^{11}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^5(Y^{11}; \mathbb{Z}) & \xrightarrow{\delta^*} & H^6(Z^{12}, Y^{11}; \mathbb{Z}) & \xrightarrow{j^*} & H^6(Z^{12}; \mathbb{Z}) & \xrightarrow{i^*} & H^6(Y^{11}; \mathbb{Z}) & \xrightarrow{\delta^*} & H^7(Z^{12}; \mathbb{Z}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Fr}^5(Y^{11}) & \xrightarrow{\bar{\delta}} & \text{Fr}^6(Z^{12}, Y^{11}) & \xrightarrow{\bar{j}} & \text{Fr}^6(Z^{12}) & \xrightarrow{\bar{i}} & \text{Fr}^6(Y^{11}) & \xrightarrow{\bar{\delta}} & \text{Fr}^7(Z^{12}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0 & & 0
 \end{array} \quad (3.6)$$

The maps  $i$  and  $\delta$  are adjoints of each other, and so are the maps  $\bar{i}$  and  $\bar{\delta}$ , with respect to the pairings that we define in the following sections. We will study the cases in ten, eleven, and twelve dimensions in more detail.

### 3.1 Middle cohomology of closed 10-manifolds

We now consider the degree five cohomology, as appropriate for the 5-form in type IIB string theory on  $X^{10}$ .

**The antisymmetric form over  $\mathbb{R}$  of a closed 10-dimensional manifold.** Consider a closed oriented 10-manifold  $X^{10}$  with (co)homology with real coefficients. In this case the intersection form  $\phi_X$  is defined using the fundamental class  $[X^{10}] \in H_{10}(X^{10}; \mathbb{R})$  and is given by

$$\phi_X : (x, y) \mapsto \langle x \cup y, [X^{10}] \rangle, \quad \text{for } x, y \in H^5(X^{10}; \mathbb{R}). \quad (3.7)$$

The fact that  $X^{10}$  is closed implies that the intersection form  $\phi_X$  is nonsingular.

**Classification of symplectic forms over  $\mathbb{R}$ .** It is natural to ask what possible intersection pairings on  $X^{10}$  can occur. These are characterized as follows

1. Every symplectic form  $(V, \phi)$  over  $\mathbb{R}$  is isomorphic to  $\mathbb{H}_-(\mathbb{R}^p) \oplus \bigoplus_r (\mathbb{R}, 0)$  with  $2p+r = \dim_{\mathbb{R}} V$ . This form is nonsingular if and only if  $r = 0$ .
2. Two forms are isomorphic if and only if they have the same  $p$  and  $r$ .
3. Every nonsingular symplectic form  $(V, \phi)$  admits a Lagrangian (as can be shown by induction on  $\dim_{\mathbb{R}} V$ ). This implies that there are always cohomology classes whose pairing with every other class is zero.

**Example 1.** The intersection form of  $X^{10} = S^5 \times S^5$ , the product of two 5-spheres, is the hyperbolic form  $\mathbb{H}_-(\mathbb{R}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . This example corresponds to  $p = 1$  and  $r = 0$  in the above classification.

Let  $Q$  be the intersection form over  $\mathbb{Z}$  and let  $b_5 = \dim H^5(X^{10}; \mathbb{R})$ . Then there exist  $b_5 \times b_5$  matrices  $A$  and  $B$  over  $\mathbb{Z}$  for which  $A^T B A = I_{b_5}$  is the identity matrix. Therefore,  $\det Q = \pm 1$ . The free abelian group  $H^5(X^{10}; \mathbb{Z})/T^5$ , where  $T^5$  is the torsion subgroup of the integral cohomology group  $H^5(X^{10}; \mathbb{Z})$ , has a basis  $\{x_1, \dots, x_{b_5}\}$  such that  $x_i \cup x_j = \delta_{ij} \Lambda$ , where  $\Lambda$  is a generator of the group  $H^{10}(X^{10}; \mathbb{Z})$ .

### 3.2 Intersection pairing on twelve-manifolds with boundary

The main operator we consider in the 12-dimensional case is the signature operator  $\mathcal{S}$ . We now provide the setting and recall some of the basic properties that are relevant to our problem, expanding on the remarks at the beginning of Section 3. For more background and details see Ref. [25] for the closed case and Ref. [5] for the case with boundary.

Let  $Z^{12}$  be a compact oriented twelve-manifold. Let  $[\omega_1]$  and  $[\omega_2]$  be elements of the middle cohomology group  $H^6(Z^{12}; \mathbb{Z})$ . The Hodge  $*$ -operator satisfies  $*^2 = 1$  when acting on a 6-form in



a twelve-manifold  $Z^{12}$ , and hence  $*$  has eigenvalues  $\pm 1$ . The signature can be identified with the signature of the intersection pairing on  $H^6(Z; \mathbb{R})$ ; if we represent cohomology classes  $x$  and  $y$  by closed forms  $\alpha$  and  $\beta$  then the intersection pairing is  $\langle x, y \rangle = \int_Z \alpha \wedge \beta$ . Comparing with  $\int_Z \alpha \wedge * \beta$ , the  $L^2$ -inner product of  $\alpha$  and  $\beta$ , we see that the intersection pairing is positive definite on the  $+1$ -eigenspace of  $*$  and negative definite on the  $-1$ -eigenspace. Indeed, consider the bilinear form on middle cohomology  $\phi_Z : H^6(Z^{12}; \mathbb{R}) \times H^6(Z^{12}; \mathbb{R}) \rightarrow \mathbb{R}$ , defined by  $\phi([\omega_1], [\omega_2]) := \int_{Z^{12}} \omega_1 \wedge \omega_2$ . This has the following properties

1.  $\phi_Z$  is a  $b_6 \times b_6$  symmetric matrix, where  $b_6 = \dim H^6(Z^{12}; \mathbb{R})$ .
2.  $\phi_Z$  is nondegenerate since  $\phi([\alpha], [\beta]) = 0$  for an  $[\alpha] \in H^6(Z^{12}; \mathbb{R})$  implies  $[\beta] = 0$ .
3. The definition of  $\phi_Z$  is independent on the representatives of  $[\omega_1]$  and  $[\omega_2]$ .
4. Poincaré duality implies that  $\phi_Z$  has maximal rank.
5. On  $Z^{12}$ ,  $\phi_Z$  has real eigenvalues,  $b_6^+$  of which are positive and  $b_6^-$  of which are negative, with  $b_6^+ + b_6^- = b_6$ . The Hirzebruch signature is defined as  $\sigma(Z^{12}) := b_6^+ - b_6^-$ .

Let  $\text{Harm}^6(Z^{12})$  be the set of harmonic 6-forms on  $Z^{12}$ . Note that  $\text{Harm}^6(Z^{12}) \cong H^6(Z^{12}; \mathbb{R})$  and each element of  $H^6(Z^{12}; \mathbb{R})$  has a unique harmonic representative. There is a corresponding splitting of  $\text{Harm}^6(Z^{12})$  into  $\pm 1$ -eigenspaces  $\text{Harm}^6(Z^{12}) = \text{Harm}_+^6(Z^{12}) \oplus \text{Harm}_-^6(Z^{12})$ , which block-diagonalizes  $\sigma$ ; indeed for  $\omega_6^\pm \in \text{Harm}_\pm^6(Z^{12})$ ,  $\phi_Z(\omega_6^+, \omega_6^+) = \int_{Z^{12}} \omega_6^+ \wedge \omega_6^+ = \int_{Z^{12}} \omega_6^+ \wedge * \omega_6^+ = (\omega_6^+, \omega_6^+) > 0$ , where  $(\omega_6^+, \omega_6^+)$  is the standard positive definite inner product on differential forms. Similarly,  $\phi_Z(\omega_6^-, \omega_6^-) = - \int_{Z^{12}} \omega_6^- \wedge * \omega_6^- = -(\omega_6^-, \omega_6^-) < 0$ , and  $\phi_Z(\omega_6^+, \omega_6^-) = - \int_{Z^{12}} \omega_6^+ \wedge * \omega_6^- = - \int_{Z^{12}} \omega_6^- \wedge * \omega_6^+ = -(\omega_6^-, \omega_6^+) = 0$ . Hence  $\phi_Z$  is block-diagonal with respect to  $\text{Harm}_+^6(Z^{12}) \oplus \text{Harm}_-^6(Z^{12})$ . Moreover,  $b_6^\pm = \dim_{\mathbb{R}} \text{Harm}_\pm^6(Z^{12})$ . Now  $\sigma(Z^{12})$  is expressed as

$$\sigma(Z^{12}) = \dim \text{Harm}_+^6(Z^{12}) - \dim \text{Harm}_-^6(Z^{12}) . \quad (3.8)$$

**Example 2: Kähler manifolds.** If a compact Kähler manifold  $Z$  is of even complex dimension, e.g. 6, then the intersection pairing on the middle cohomology is of the form  $\text{sign}(Z) = \sum_{p,q=0}^6 (-1)^p h^{p,q}(Z)$ , where  $h^{p,q}(Z) := \dim H^{p,q}(Z)$  are the Hodge numbers.

**The signature index theorem.** Poincaré duality shows that the Euler characteristic is given by  $\chi(Z^{12}) = b_6 \bmod 2$ , so that  $\sigma(Z^{12}) = \chi(Z^{12}) \bmod 2$ . Consider the operator  $\mathcal{S} := d + d^\dagger = d + *d*$ . Since the (Hodge) Laplacian  $\Delta = \mathcal{S}^2$  is self-dual on  $\Omega^*(Z^{12})$ , the index of  $\Delta$  vanishes identically. Also,  $\mathcal{S}$  is self-adjoint,  $\mathcal{S} = \mathcal{S}^\dagger$ , on forms  $\Omega^*(Z^{12})$  and so  $\text{ind}(\mathcal{S}) = 0$  as well. However, a nontrivial complex is obtained when restricting  $\mathcal{S}$  to even forms;  $\mathcal{S}^{\text{ev}} : \Omega^{\text{ev}}(Z^{12})^{\mathbb{C}} \rightarrow \Omega^{\text{ev}}(Z^{12})^{\mathbb{C}}$ , where  $\Omega^{\text{ev}}(Z^{12})^{\mathbb{C}} := \bigoplus_i \Omega^{2i}(Z^{12})^{\mathbb{C}}$  and  $\Omega^{\text{odd}}(Z^{12})^{\mathbb{C}} := \bigoplus_i \Omega^{2i+1}(Z^{12})^{\mathbb{C}}$ . The adjoint operator is  $\mathcal{S}^{\text{odd}} := \mathcal{S}^{\text{ev}\dagger} : \Omega^{\text{odd}}(Z^{12})^{\mathbb{C}} \rightarrow \Omega^{\text{ev}}(Z^{12})^{\mathbb{C}}$ . Then the corresponding kernels are given by even and odd harmonic forms  $\ker(\mathcal{S}^{\text{ev}}) = \bigoplus \text{Harm}^{2i}(Z^{12})$ ,  $\ker(\mathcal{S}^{\text{odd}}) = \bigoplus \text{Harm}^{2i+1}(Z^{12})$ , respectively. As a result, the index calculates the Euler characteristic

$$\text{ind}(\mathcal{S}^{\text{ev}}) = \dim \ker(\mathcal{S}^{\text{ev}}) - \dim \ker(\mathcal{S}^{\text{odd}}) = \chi(Z^{12}) . \quad (3.9)$$

For a complex-valued  $r$ -form  $\omega \in \Omega^r(Z^{12})^{\mathbb{C}}$ , application of the Hodge operator twice gives  $**\omega = (-1)^r \omega$ . Define a square root via the operator  $\pi : \Omega^r(Z^{12})^{\mathbb{C}} \rightarrow \Omega^{12-r}(Z^{12})^{\mathbb{C}}$  given by  $\pi := i^{r(r-1)+6} *$  and which anticommutes with  $\mathcal{S}$ ,  $\{\pi, \mathcal{S}\} = \pi\mathcal{S} + \mathcal{S}\pi = 0$ . Let  $\pi$  act on  $\Omega^*(Z^{12})^{\mathbb{C}} = \Omega^r(Z^{12})^{\mathbb{C}}$ . Since

$\pi^2 = 1$ , the eigenvalues of  $\pi$  are  $\pm 1$ . This gives a decomposition of  $\Omega^*(Z^{12})^{\mathbb{C}}$  into the  $\pm 1$ -eigenspaces  $\Omega^{\pm}(Z^{12})$  of  $\pi$  as  $\Omega^*(Z^{12})^{\mathbb{C}} = \Omega^+(Z^{12}) \oplus \Omega^-(Z^{12})$ . Since  $\mathcal{S}$  anticommutes with  $\pi$ , the restriction of  $\mathcal{S}$  to  $\Omega^+(Z^{12})$  defines the signature complex  $\mathcal{S}_+ : \Omega^+(Z^{12}) \rightarrow \Omega^-(Z^{12})$ , where  $\mathcal{S}_+ := \mathcal{S}|_{\Omega^+(Z^{12})}$ . The index of the signature complex is

$$\begin{aligned} \text{ind} \mathcal{S}_+ &= \dim \ker(\mathcal{S}_+) - \dim \ker(\mathcal{S}_-) \\ &= \dim \text{Harm}(Z^{12})^+ - \dim \text{Harm}(Z^{12})^-, \end{aligned}$$

where  $\mathcal{S}_- := \mathcal{S}^{\dagger} : \Omega^-(Z^{12}) \rightarrow \Omega^+(Z^{12})$  and  $\text{Harm}(Z^{12})^{\pm} := \{\omega \in \Omega^{\pm}(Z^{12}) \mid \mathcal{S}_{\pm} \omega = 0\}$ . Note that  $\text{Harm}^6(Z^{12})^{\pm} = \text{Harm}_{\pm}^6(Z^{12})$  since  $\pi = *$  in  $\text{Harm}^6(Z^{12})$ . The index theorem is [25]

$$\text{ind} \mathcal{S}_+ = \sigma(Z^{12}) = \int_{Z^{12}} [L(TZ^{12})]_{(12)} . \quad (3.10)$$

**Classification of symmetric forms over  $\mathbb{R}$ .** As in the antisymmetric 10-dimensional case, it is natural to ask which possible intersection forms might arise in the 12-dimensional case. These can be characterized as follows:

1. Every symmetric form  $(V, \phi)$  is isomorphic to the direct sum  $\bigoplus_p(\mathbb{R}, 1) \oplus \bigoplus_q(\mathbb{R}, -1) \oplus \bigoplus_r(\mathbb{R}, 0)$  with  $p + q + r = \dim_{\mathbb{R}} V$ . The form  $(V, \phi)$  is nonsingular if and only if  $r = 0$ .
2. Two forms are isomorphic if and only if they have the same nonnegative integers  $p, q$ , and  $r$ .
3. The signature or index of  $(V, \phi)$  is  $\sigma(V, \phi) = p - q \in \mathbb{Z}$ .
4. The following three conditions on a nonsingular forms  $(V, \phi)$  are equivalent
  - (a)  $\sigma(V, \phi) = 0$ , that is  $p = q$  (split signature).
  - (b)  $(V, \phi)$  admits a Lagrangian  $L$ .
  - (c)  $(V, \phi)$  is isomorphic to  $\bigoplus_p(\mathbb{R}, 1) \oplus \bigoplus_p(\mathbb{R}, -1) \cong \mathbb{H}_+(\mathbb{R}^p)$ .

**Example 3.** The intersection form of the product of two 6-spheres,  $Z^{12} = S^6 \times S^6$ , is the symmetric hyperbolic form  $(H^6(S^6 \times S^6; \mathbb{R}), \phi_Z) = \mathbb{H}_+(\mathbb{R}) = (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ . Consequently, the signature is  $\sigma(S^6 \times S^6) = \sigma(\mathbb{H}(\mathbb{R})) = 0$ . This corresponds to the values  $p = 2$ ,  $q = 0$ , and  $r = 0$ , in the above classification.

**The APS index for the case when  $Z^{12}$  has nonempty boundary.** Consider the signature operator  $\mathcal{S}$  on the 12-manifold  $Z^{12}$  when  $\partial Z^{12} = Y^{11}$  is nonempty. In this case, in addition to the Hirzebruch L-genus, one gets the corresponding eta invariant via the APS index theorem [5]

$$\text{ind}(\mathcal{S}_+) = \int_{Z^{12}} [L(TZ^{12})]_{(12)} - \eta_{\mathcal{S}} . \quad (3.11)$$

Unlike the characteristic form  $L$ , the invariant  $\eta_{\mathcal{S}}$  is an analytic invariant. However, there are geometric ways of calculating this invariant without full knowledge of the spectrum of the operator (see [45] for a recent review in the related context of M-theory).

**The symmetric form of a 12-manifold with boundary over  $\mathbb{R}$ .** Consider an oriented 12-dimensional manifold  $Z^{12}$  with boundary  $\partial Z = Y^{11}$  with (co)homology taken with real coefficients. The intersection form of  $(Z^{12}, Y^{11})$  is the symmetric form given by the evaluation of the cup product on the fundamental class  $[Z^{12}] \in H_{12}(Z^{12}, Y^{11}; \mathbb{R})$ ,

$$\phi_Z : (x, y) \mapsto \langle x \cup y, [Z^{12}] \rangle \quad \text{for } x, y \in H^6(Z^{12}, Y^{11}; \mathbb{R}). \quad (3.12)$$

Poincaré duality and the universal coefficient theorem imply the relation between homology and cohomology in degree six

$$H^6(Z^{12}, Y^{11}; \mathbb{R}) \cong H_6(Z^{12}; \mathbb{R}), \quad H^6(Z^{12}, Y^{11}; \mathbb{R}) \cong H_6(Z^{12}, Y^{11})^*. \quad (3.13)$$

These groups fit into an exact sequence

$$\cdots \rightarrow H_6(Y^{11}; \mathbb{R}) \rightarrow H_6(Z^{12}; \mathbb{R}) \xrightarrow{\phi_Z} H_6(Z^{12}, Y^{11}; \mathbb{R}) \rightarrow H_5(Y^{11}; \mathbb{R}) \rightarrow \cdots. \quad (3.14)$$

The isomorphism class of the intersection form is a homotopy invariant of  $(Z^{12}, Y^{11})$ .

**Example 4.** Take  $Z^{12} = \mathbb{D}^6 \times S^6$ . This manifold has a boundary  $\partial Z = Y^{11} = S^5 \times S^6$ , and then from the above relations we have  $H^6(\mathbb{D}^6 \times S^6, S^5 \times S^6; \mathbb{R}) \cong H_6(\mathbb{D}^6 \times S^6; \mathbb{R})$ , as expected.

Next we consider more fully the relation between homology and cohomology in our context in the two other relevant dimensions, i.e. dimensions ten and eleven.

### 3.3 Cohomology vs. homology

For the purpose of connecting to geometry, and in particular for considering the action of the diffeomorphism group, it will be more convenient to work with homology rather than with cohomology. In this section we study how relevant properties of one group get translated to the other. We make use of basic properties that can be found e.g. in [42]. The 12-dimensional case was considered towards the end of the previous section, so we concentrate on the 10-dimensional case and also briefly on the 11-dimensional case.

**(Co)homology as a module  $V$  over a field.** Let  $M^{2n}$  be a closed oriented manifold and  $\mathbb{F}$  an arbitrary field. Then  $H_n(M^{2n}; \mathbb{F})$  is an inner product space over  $\mathbb{F}$ , using the intersection number as inner product. The latter is either symmetric (for  $n$  even) or antisymmetric (for  $n$  odd). Two cases are of particular interest to us:

1. *For  $\mathbb{F} = \mathbb{Z}_2$ , mod 2 coefficients:* If  $x, y \in H_n(M^{2n}; \mathbb{Z}_2)$  the intersection number is symmetric  $\phi_M^*(x, y) = \phi_M^*(y, x) \in \mathbb{Z}_2$ . Poincaré duality implies that the homology  $H_n(M^{2n}; \mathbb{Z}_2)$  is an inner product space over  $\mathbb{Z}_2$ .
2. *For  $\mathbb{F} = \mathbb{Z}$ , integral coefficients:* The  $\mathbb{Z}$ -module  $\text{Fr}_n(M^{2n}) = H_n(M^{2n}; \mathbb{Z}) / \{\text{torsion subgroup}\}$  is an inner product space over  $\mathbb{Z}$ .

We start with ten dimensions.

**Dimension of homology.** The matrix of the intersection form of the manifold  $X^{10}$  is antisymmetric, and hence of even rank. Since the matrix is nondegenerate, the rank should be equal to  $\dim H_5(X^{10}; \mathbb{R})$ . Therefore, this dimension is even. In fact, this also follows from the fact that the Euler characteristic  $\chi(X^{10})$  is even and that it has the same mod 2 value as the dimension of the middle cohomology.

**Intersection pairings.** Let  $X^{10}$  be a closed oriented 10-manifold. The intersection pairing on homology  $\phi_X^* : H_i(X^{10}) \otimes H_{10-i}(X^{10}) \rightarrow \mathbb{Z}$  is given by  $\phi_X^*(\alpha, \beta) = \langle PD^{-1}(\alpha), \beta \rangle$ , where  $PD^{-1}$  is the inverse of the Poincaré duality isomorphism. Consider the restriction of  $\phi_X^*$  to the free module  $\text{Fr}_5(X^{10}) = H_5(X^{10}; \mathbb{Z})/\text{Torsion}$ . If we choose a basis for  $\text{Fr}_5(X^{10})$  then the intersection pairing is represented by an antisymmetric matrix whose determinant is  $\pm 1$ , i.e. is a unimodular matrix. Such a pairing is called perfect.

We will concentrate on the case  $i = 5$  and work with more general coefficients. Let  $\mathbb{F}$  be any field. On the space  $H^5(X^{10}; \mathbb{F})$  we have seen that there is the bilinear form  $\phi_X(x, y) = \langle x \cup y, [X^{10}] \rangle$ , where  $[X^{10}]$  is the fundamental class in  $H_{10}(X^{10}; \mathbb{F})$ . For the opposite orientation on  $X^{10}$ , i.e. taking  $-X^{10}$ , the fundamental class changes sign  $[-X^{10}] = -[X^{10}]$ . Therefore, the bilinear form  $\phi_X$  changes sign as well:  $\phi_{-X} = \phi_X$ , where  $\phi_{-X}$  is the bilinear form for  $-X^{10}$ . On the dual space  $H_5(X^{10}; \mathbb{F})$  there is the dual form  $\phi_X^*(\alpha, \beta) = \langle \langle \alpha, \beta \rangle \rangle$ , the intersection number. This also depends on the orientation as in the case for cohomology.

**Relating integral homology and cohomology.** The universal coefficient theorem relates the homology groups  $H_5(X^{10}; \mathbb{R}) \cong H_5(X^{10}; \mathbb{Z}) \otimes \mathbb{R} \cong (H_5(X^{10}; \mathbb{Z})/T_5) \otimes \mathbb{R}$ . Let us consider this in more generality. Let  $T_k(M)$  denote the torsion submodule of  $H_k(M; \mathbb{Z})$ , i.e.

$$T_k(M) = H_k(M; \mathbb{Z})_{\text{tors}} = \{ \alpha \in H_k(M; \mathbb{Z}) \mid r\alpha = 0 \text{ for some } r \in \mathbb{Z} \} . \quad (3.15)$$

Choose a complement  $\text{Fr}_k(M)$  of  $T_k(M)$  in  $H_k(M; \mathbb{Z})$ , i.e. a free submodule of  $H_k(M; \mathbb{Z})$  so that  $H_k(M; \mathbb{Z}) \cong \text{Fr}_k(M) \oplus T_k(M)$ . Applying the universal coefficient theorem with  $G = \mathbb{Z}$  gives the (noncanonical) isomorphisms

$$H^k(M; \mathbb{Z}) \cong \text{Fr}_k(M) \oplus T_{k-1}(M) . \quad (3.16)$$

Note that the integral cohomology not only depends on the free part of the homology in that degree but also, interestingly, on the torsion shifted down by one degree. We will make use of this in section 4.

Now if we take  $M$  to be an oriented  $m$ -manifold, then there is the Poincaré duality isomorphism  $H_k(M; \mathbb{Z}) \cong H^{m-k}(M; \mathbb{Z})$ . Combining with the above symmetries gives the isomorphisms

$$\text{Fr}_k(M) \cong \text{Fr}_{m-k}(M) , \quad T_k(M) \cong T_{m-k-1}(M) . \quad (3.17)$$

Hence, for  $m = 10$ ,  $k = 5$  we have the cohomology in terms of homology relation for  $X^{10}$

$$H^5(X^{10}; \mathbb{Z}) \cong \text{Fr}_5(X^{10}) \oplus T_4(X^{10}) . \quad (3.18)$$

Therefore, we observe that if  $H_4(X^{10}; \mathbb{Z})_{\text{tors}} = 0$  then middle integral cohomology is isomorphic to the free part of the integral middle homology. If this happens then there would be no torsion ( $p, q$ ) D3-branes. In the presence of such branes, however, one has to deal with torsion 4-cycles.

**Example 5: Torsion in homology of degree four.** In light of equation (3.18), we need to get some idea about the torsion  $T_4(X^{10})$  in  $H_4(X^{10}; \mathbb{Z})$ . Alternatively, we can look at  $H_4(X^{10}; \mathbb{Z}_p)$ , that is degree-four homology with coefficients in the cyclic group  $\mathbb{Z}_p$  for  $p$  a prime. If  $H_4(X^{10}; \mathbb{Z})$  and  $H_3(X^{10}; \mathbb{Z})$  are both finitely generated, e.g. if we take them of the form

$$H_4(X^{10}; \mathbb{Z}) \supset a(\mathbb{Z}) \oplus b(\mathbb{Z}_{p^k}), \quad H_3(X^{10}; \mathbb{Z}) \supset c(\mathbb{Z}) \oplus d(\mathbb{Z}_{p^k}), \quad k \geq 1, \quad (3.19)$$

then the universal coefficient theorem can be used (see [22]) to show that

$$H_4(X^{10}; \mathbb{Z}_p) \cong (a + b + d)\mathbb{Z}_p. \quad (3.20)$$

**Torsion and intersection forms.** As we saw above, the bilinear forms  $\phi_X$  and  $\phi_X^*$  can be defined not only over a field  $\mathbb{F}$  but also over the ring  $\mathbb{Z}$ . In this case,  $H_n(X^{10})$  must be replaced by the free abelian group  $H_5(X^{10}; \mathbb{Z})/T_5$ , where  $T_5$  is the torsion subgroup, because the intersection form vanishes on elements of finite order. The elements of finite order do not affect the intersection numbers: if  $\alpha, \beta \in C_5(X^{10}; \mathbb{Z})$  and  $r\alpha, s\beta \in C_5(X^{10}; \mathbb{R})$  are cycles, then the intersection forms are related as  $\langle r\alpha, s\beta \rangle = rs\langle \alpha, \beta \rangle$ . Therefore, the intersection forms over  $\mathbb{R}$  and  $\mathbb{Z}$  have the same matrix. This implies, in particular, that  $H_5(X^{10}; \mathbb{R})$  has a basis in which the intersection form has integer coefficients. The significance of this for us is that torsion in ten dimensions will not need to be considered. In fact, later when studying diffeomorphisms we will assume  $H_5(X^{10}; \mathbb{Z})$  to be torsion-free, i.e.  $T_5(X^{10}) = 0$ . However, we will see that is far from being the case in eleven dimensions.

### 3.4 Quadratic forms and their refinements

In this section we will consider quadratic refinements of the intersection forms that we encountered in the previous sections, mainly in Section 3.1 and Section 3.2. We start with the motivation and then go through a more detailed (and formal) description.

**Quadratic functions from type IIB.** The construction of the partition function for the self-dual 5-form field in type IIB string theory requires the existence of a function  $\Omega(x)$  from  $H^5(X^{10}; \mathbb{Z})$  to the group  $\mathbb{Z}_2 = \{\pm 1\} \subset U(1)$  obeying, for all  $x, y \in H^5(X^{10}; \mathbb{Z})$ , the relation [57]

$$\Omega(x + y) = \Omega(x)\Omega(y)(-1)^{x \cdot y}, \quad (3.21)$$

where  $x \cdot y$  is the intersection pairing  $\int_{X^{10}} x \cup y$ . Furthermore, if we write  $\Omega(x) = (-1)^{h(x)}$ , then the mod 2 number  $h(x)$  is given by  $h(x) = \int_{Z^{12}} z \cup x$ , where  $z$  is a degree six cohomology class in  $H^6(Z^{12}; \mathbb{Z})$ , extending  $x$ , with  $Z^{12}$  the bounding Spin 12-manifold of the extension  $Y^{11}$  of  $X^{10}$  by a circle. When  $Z^{12}$  is Spin,  $h(x)$  is always even, so that there is no refinement, and hence no ambiguities in the partition function. However, Witten points out that it is more convenient to take  $Z^{12}$  to be only oriented and not necessarily Spin. In this case,  $h(x)$  is no longer necessarily well-defined mod 2, and the remedy for this is to replace the expression for  $h(x)$  by  $\int_{Z^{12}} (z \cup x + v_6 \cup x)$ , which is always even. Here  $v_6$  is the 6th Wu class of  $Z^{12}$  (see section 3.6 and section 3.7). If  $z$  is taken to be a pull back from  $Y^{11} = \partial Z^{12}$  then  $z \cup x$  vanishes for dimensional reasons near the boundary, and the second summand also vanishes near the boundary because of the Spin condition  $w_2 = 0$  there. This is put on firm mathematical ground by Hopkins and Singer [26].

**Remarks.** We state a few comments to help us proceed with the discussion.

1. The function  $\Omega(x)$  above is written multiplicatively, i.e. using multiplication instead of addition. We note that when written additively, it coincides with the usual quadratic function, with the rule

$$q(x + y) = q(x) + q(y) + \phi_X(x, y) \quad (3.22)$$

replacing (3.21), and with  $\Omega$  replaced by  $q$ .

2. The above analysis for the partition function requires a circle bundle, an instance of which is the product  $Y^{11} = X^{10} \times S^1$ . In comparison to our setting, this corresponds to the special case of the mapping torus with identity diffeomorphism.

3. We will generally work with Wu-oriented manifolds.

4. We will consider the relationship between classes on  $Z^{12}$ , classes on  $Y^{11}$  and classes on  $X^{10}$ . This will be done both for ‘general’ classes such as  $z$  as well as ‘specific’ classes such as the Wu class  $v_6$ .

5. In Ref. [7] an approach was taken by looking at the bounding 11-manifold to  $X^{10}$  in order to study the partition function of the self-dual theory. There, a choice of solution  $\Omega$  is referred to as a choice of QRIF (Quadratic Refinement of the Intersection Form). What we do here instead is a Chern-Simons construction in the sense of circle bundle then bounding.

**Quadratic and bilinear forms.** Let  $V$  be a finite-dimensional vector space over a field  $\mathbb{F}$ . A *quadratic form* on  $V$  is a map  $q : V \rightarrow \mathbb{F}$  satisfying

1. *Homogeneity in degree two:*  $q(ax) = a^2q(x)$  for all  $x$  in  $V$  and  $a$  in  $\mathbb{F}$ .
2. *Polar identity:* The map  $\varphi_q : V \times V \rightarrow \mathbb{F}$ , defined by  $\varphi_q(x, y) = q(x + y) - q(x) - q(y)$ , is a bilinear form. This is called the polar form of  $q$ . Note that if  $\mathbb{F}$  has characteristic 2 then the polar form is automatically symmetric.

The above relation in the second property can be ‘inverted’ to give  $q$  in terms of  $\varphi$ . We start with a bilinear form  $\varphi : V \times V \rightarrow \mathbb{F}$  is a bilinear form, and let  $q_\varphi : V \rightarrow \mathbb{F}$  be defined by  $q_\varphi(x) = \varphi(x, x)$  for all  $x$  in  $V$ . Then  $q_\varphi$  is a quadratic form with polar form  $\varphi_{q_\varphi} = \varphi + \varphi^T$ .

**Working in a basis.** Let  $B = \{e_1, \dots, e_n\}$  be an ordered basis for  $V$ . Then elements  $x, y$  in  $V$  have coordinate components  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  in the basis  $B$ , and the bilinear form in this basis is

$$\varphi(x, y) = \varphi(x_1e_1 + \dots x_ne_n, y_1e_1 + \dots + y_ne_n) = \sum_{i,j} \varphi(e_i, e_j)x_ix_j. \quad (3.23)$$

Then the matrix  $[\varphi]_B := (\varphi(e_i, e_j))$  on a given ordered basis completely determines the bilinear form. Consequently, in matrix notation, we write  $\varphi(x, y) = [x]^T [\varphi]_B [y]$ . Two bilinear forms  $\varphi$  and  $\varphi'$  are isomorphic if and only if  $[\varphi']_B = A[\varphi]_B A^T$  for some matrix  $A \in GL(n, \mathbb{Z})$ .

**Symmetric bilinear forms.** If  $\varphi$  is a symmetric bilinear form then the quadratic form associated to  $\varphi$  is the function  $q : V \rightarrow \mathbb{Z}$  defined by  $q(x) = \varphi(x, x)$ . A bilinear form over  $\mathbb{Z}$  is called *even* (or *type II*) if  $\varphi(x, x)$  is even for all  $x$  in  $V$ . Since  $\varphi(x + y, x + y) = \varphi(x, x) + \varphi(y, y) + 2\varphi(x, y)$ , the

bilinear form is even if and only if all elements on the diagonal, in the matrix description, are even. Note that, except in the case of characteristic 2, there is always an ordered basis for  $V$  in which  $\varphi$  is represented by a diagonal matrix. A symmetric basis for  $q$ , or  $\varphi_q$ , is a basis  $e_1, \dots, e_n$  such that the associated matrix  $\varphi_q(e_i, e_j)$  has the generalized symmetric hyperbolic form  $\mathbb{H}_+ = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ .

**Example 6.** Let  $a, b \in \mathbb{F}$ . Consider the 2-dimensional quadratic form on  $\mathbb{F} \times \mathbb{F}$  given by  $q(x, y) = ax^2 + xy + by^2$ . The corresponding matrix for  $q$  in the standard basis is  $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ , while the corresponding matrix for the polar form  $\varphi_q$  is  $\begin{pmatrix} 2a & 1 \\ 1 & 2b \end{pmatrix} = A + A^T$ .

**Isotropic bilinear forms.** A bilinear form  $\varphi : V \times V \rightarrow \mathbb{F}$  is *isotropic* if  $\varphi(x, x) = 0$  for all  $x$  in  $V$ . Note that  $\varphi(x, y) + \varphi(y, x) = \varphi(x + y, x + y) - \varphi(x, x) - \varphi(y, y) = 0$ , so that every isotropic form is antisymmetric. The converse is not true in general. However, for  $\mathbb{F} = \mathbb{Z}_2$ , the converse holds since having  $\varphi(x, x) = -\varphi(x, x)$  implies that  $\varphi(x, x) = 0$ . In fact, if  $\mathbb{F} = \mathbb{Z}_2$  the bilinear form  $\varphi$  is necessarily isotropic and it is always the case that  $V$  possesses a symmetric basis. The first part of the fact can be seen from  $\varphi_q(x, x) = q(2x) - 2q(x) = 0$ , since  $2x = 0 \in V$  and  $2q(x) = 0 \in \mathbb{Z}_2$ .

### 3.5 Quadratic forms on homology over $\mathbb{Z}_2$ and $\mathbb{Q}/\mathbb{Z}$ and the Arf invariant

We have seen (cf. expression (3.21)) that the quadratic functions in type IIB string theory in ten dimensions take values in  $\mathbb{Z}_2$ . On the other hand, in eleven dimensions the relevant forms take values in  $\mathbb{Q}/\mathbb{Z}$  (cf. expression (3.5)). In this section we provide further characterization of such forms.

We consider a 10-dimensional Spin manifold  $X^{10}$  and form the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$ , which is an 11-dimensional Spin manifold. Then we form the bounding twelve-dimensional manifold  $Z^{12}$ . We will consider the middle-dimensional homology of  $X^{10}$  and study the corresponding ‘lifts’ to  $Y^{11}$  and to  $Z^{12}$ . We will also investigate what happens to the intersection pairing in the process. This is a homological analog of the discussion in Section 3.1 and Section 3.2.

#### 3.5.1 Quadratic forms in ten dimensions

Consider  $X^{10}$ , a closed Spin 10-manifold. Poincaré duality on homology with  $\mathbb{Z}_2$  coefficients gives a nonsingular symmetric (since over  $\mathbb{Z}_2$ ) bilinear pairing

$$\phi_X^* : H_5(X^{10}; \mathbb{Z}_2) \otimes H_5(X^{10}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2. \quad (3.24)$$

Using the construction in Ref. [12] we can define a quadratic refinement  $q : H_5(X^{10}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  of the  $\mathbb{Z}_2$ -intersection pairing  $\phi_X^*$  which is essentially unique. Hence we can associate to each Spin manifold  $(X^{10}, \omega)$ , with Spin structure  $\omega$ , the Arf invariant  $\text{Arf}(q)$  of  $q$  called the *generalized Kervaire invariant*.

**The Arf invariant.** Let  $\alpha_i, \beta_i$ , for  $i = 1, \dots, n$ , be a symmetric basis for  $q$ , i.e.

$$\varphi_q(\alpha_i, \beta_i) = \delta_{ij}, \quad \varphi_q(\alpha_i, \alpha_j) = \varphi_q(\beta_i, \beta_j) = 0. \quad (3.25)$$

Then the *Arf invariant* is defined as

$$A_q = \sum_{i=1}^n q(\alpha_i)q(\beta_i) \in \mathbb{Z}_2. \quad (3.26)$$

If  $B = \{e_1, \dots, e_n\}$  is a basis for the vector space  $V$ , then any matrix  $M$  such that  $q(x) = x^T M x$  is called a matrix of  $q$  with respect to  $B$ . There is more than one possibility for the matrix  $M$ , but in the upper triangular form it is uniquely determined and given by the *normal form*  $M = (m_{ij})$  with entries  $m_{ij}$  equal to  $q(e_i)$  for  $i = j$ , to  $\varphi_q(e_i, e_j)$  for  $i < j$  (above the diagonal), and to 0 otherwise; that is we have the block-diagonal form  $M = \begin{pmatrix} A & I_n \\ O_n & B \end{pmatrix}$ , where  $A = \text{diag}(q(\alpha_1), \dots, q(\alpha_n))$ ,  $B = \text{diag}(q(\beta_1), \dots, q(\beta_n))$ ,  $I_n$  is the  $n \times n$  identity matrix, and  $O_n$  is the  $n \times n$  zero matrix. The Arf invariant can then be read off as  $A_q = \text{trace}(AB)$ . For any other matrix of  $q$  in this same basis, say  $\begin{pmatrix} A' & C \\ D & B' \end{pmatrix}$ , we have  $\text{trace}(AB) = \text{trace}(A'B')$  so that the Arf invariant can indeed be read from any matrix of  $q$  in a symplectic basis.

**Example 7:**  $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Let  $\mathbb{H}_+$  be the hyperbolic space with matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  on the basis  $(\alpha, \beta)$ . There are two quadratic forms  $q_i : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ , compatible with this bilinear form defined over  $\mathbb{Z}_2$ , given by

$$\begin{aligned} \mathbb{H}_0 : q_0(\alpha) &= q_0(\beta) = 0, \\ \mathbb{H}_1 : q_1(\alpha) &= q_1(\beta) = 1. \end{aligned}$$

The two are manifestly not equivalent as quadratic forms. The vector space  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  has only three nontrivial elements and is generated by any two of them. Any change of basis  $B = \{\alpha, \beta\}$  to  $B' = \{\alpha', \beta'\}$  the relations  $\alpha' = \alpha$ ,  $\beta' = \alpha' + \beta'$  hold after a possible change in the order of  $\alpha$  and  $\beta$ . The new basis is still symplectic and the Arf invariant in this basis is  $A_q = q(\alpha')q(\beta')$ . Using the transformation and the fact that  $q(\alpha + \beta) = \varphi(\alpha + \beta) + q(\alpha) + q(\beta)$  and  $\varphi(\alpha, \beta) = 1$ , the Arf invariant takes the form  $A_q = q(\alpha) + [q(\alpha)]^2 + q(\alpha)q(\beta)$ . Now  $q(\alpha) + [q(\alpha)]^2 = 2q(\alpha) = 0$  in  $\mathbb{Z}_2$  so that  $A_q = q(\alpha)q(\beta)$ , demonstrating that indeed the Arf invariant is well-defined for forms on  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Since  $A_{\mathbb{H}_0} = 0$  and  $A_{\mathbb{H}_1} = 1$  and these are the only quadratic forms in two dimensions, this shows that the Arf invariant completely classifies quadratic forms in dimension two. This is not the case when the dimension of the vector space, i.e. the rank of the middle cohomology, is greater than two.

**Consequence of  $H_5(X^{10}; \mathbb{Z})$  being torsion-free.** We will be interested in considering the case when  $H_5(X^{10}; \mathbb{Z})$  is torsion-free. Then by Poincaré duality the homology group  $H_4(X^{10}; \mathbb{Z})$  would also be torsion-free. The universal coefficient theorem for homology  $H_n(M; G) \cong H_n(M; \mathbb{Z}) \otimes G \oplus \text{Tor}(H_{n-1}(M; \mathbb{Z}), G)$  implies for  $M = X^{10}$ ,  $G = \mathbb{Z}_2$  and  $n = 5$ , the isomorphism  $H_5(X^{10}; \mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{\cong} H_5(X^{10}; \mathbb{Z}_2)$ , under which the intersection pairing  $\phi_X$  induces a  $\mathbb{Z}_2$ -intersection pairing  $\phi_X \pmod{2}$ . In particular, a quadratic refinement of this  $\mathbb{Z}_2$ -intersection pairing may be identified with a map  $q : H_5(X^{10}; \mathbb{Z}) \rightarrow \mathbb{Z}_2$  such that, for all  $x, y$  in  $H_5(X^{10}; \mathbb{Z})$ ,

$$q(x + y) - q(x) - q(y) = \phi_X^*(x, y) \pmod{2}. \quad (3.27)$$

### 3.5.2 Quadratic forms in eleven dimensions

Consider the mapping torus  $(Y^{11}, \omega')$ , a compact Spin 11-manifold with Spin structure  $\omega'$ . Consider the torsion subgroup  $T_5(Y^{11})$  of the homology group  $H_5(Y^{11}; \mathbb{Z})$ . Then we have a symmetric bilinear pairing, a homological counterpart of the cohomological pairing (3.5),

$$L : T_5(Y^{11}) \otimes T_5(Y^{11}) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (3.28)$$



called the *linking pairing*, defined as follows. Given two classes  $y_1, y_2$  in  $T_5(Y^{11})$ , we represent them respectively by cycles  $\zeta_1$  and  $\zeta_2$ . Since these are torsion classes, there exists an integer  $n$  such that  $n \cdot \zeta_1$  is the boundary of a 6-chain  $\xi$ , that is  $\partial\xi = n \cdot \zeta_1$ . Define  $L(y_1, y_2)$  by the formula

$$L(y_1, y_2) = \left(\frac{1}{n}\right) \cdot (\text{intersection number of } \xi \text{ and } \zeta_2) . \quad (3.29)$$

Poincaré duality and the universal coefficient theorem imply that this symmetric pairing  $L$  is nonsingular. Corresponding to this pairing  $L$  there is, via the general construction of Ref. [13], the following quadratic refinement

$$Q_L : T_5(Y^{11}) \rightarrow \mathbb{Q}/\mathbb{Z} . \quad (3.30)$$

The *generalized Arf invariant* of  $(Y^{11}, \omega')$  is defined by

$$\text{Arf}(Y^{11}, \omega') = A(Q_L) \quad (3.31)$$

i.e. as the Arf invariant of the quadratic refinement  $Q_L$ . We will consider this invariant in the context of diffeomorphisms in section 4.

### 3.6 Characteristic vectors and signature modulo 8

In this section we provide an algebraic description of the signature modulo 8 appearing in equation (2.4), the formula for the global anomaly. The corresponding geometric aspects, together with the action of diffeomorphisms, will be discussed in section 4.

We will need the following definition. Let  $V$  be a vector space over  $\mathbb{Z}$ . An element  $v \in V$  is called *characteristic* if  $v \cdot x \equiv x \cdot x \pmod{2}$  for every  $x$  in  $V$ .

In a basis, the definition of a characteristic is equivalent to the system of congruences  $\sum_{j=1}^n a_{ij} v_j \equiv a_{ii} \pmod{2}$ , for  $i = 1, \dots, n$ , where  $(a_{ij})$  is the matrix representing the bilinear form  $\varphi$  in the given basis. We can always find a characteristic by considering the stronger system of equations  $\sum_{ij} a_{ij} v_j = a_{ii}$ , for  $i = 1, \dots, n$ . This system will always have an integral solution since  $\det(a_{ij}) = \pm 1$ , and this solution is certainly a solution to the original congruence.

**Existence of a characteristic element.** From a vector space  $V$  over  $\mathbb{Z}$  we can form the induced vector space  $V \otimes \mathbb{Z}_2$  over  $\mathbb{Z}_2$ . Let  $\bar{x}$  denote the image in  $V \otimes \mathbb{Z}_2$  of the element  $x$  in  $V$  (that is, mod 2 reduction). Then the inner product  $x \cdot y$  in  $V$  gives rise to a  $\mathbb{Z}_2$ -valued inner product  $\bar{x} \cdot \bar{y} = \{\text{residue class of } x \cdot y \pmod{2}\}$  on  $V \otimes \mathbb{Z}_2$ . Since the function  $V \otimes \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ , given by  $\bar{x} \mapsto \bar{x} \cdot \bar{x}$ , is  $\mathbb{Z}_2$ -linear there is a unique element  $\bar{v} \in V \otimes \mathbb{Z}_2$  which satisfies the equation  $\bar{v} \cdot \bar{x} = \bar{x} \cdot \bar{x}$  for all  $\bar{x}$ . Then the desired characteristic element is simply any preimage  $v$  in  $V$ . Therefore, every vector space over  $\mathbb{Z}$  possesses a characteristic element.

**Uniqueness of a characteristic element and the signature.** For any symmetric unimodular bilinear form on a lattice  $\Lambda$  (i.e. a finite abelian group), let  $\langle \cdot, \cdot \rangle$  be a symmetric unimodular pairing. Then all characteristic vectors in  $\Lambda$  have the same square modulo 8 and they are all equivalent modulo 2. Such a square is congruent to the signature modulo 8. If  $v'$  is another characteristic element for  $V$  then, by uniqueness of the mod 2 reduction,  $v'$  is necessarily of the form  $v' = v + 2x$ . Now the inner product in  $\mathbb{Z}$  of the new vector is  $v' \cdot v' = v \cdot v + 4(v \cdot x + x \cdot v)$ ,

which, upon using the definition of a characteristic element, is congruent to  $v \cdot v \pmod{8}$ . Hence the residue class of  $v \cdot v$  modulo 8 is an invariant of the vector space  $V$ . This invariant is additive with respect to direct sums. Now, for  $p$  plus entries and  $q$  minus entries, the signature is  $\sigma = p - q$ . Then if we form the orthogonal sum of  $p$  copies of the inner product space  $\langle 1 \rangle$  and  $q$  copies of the inner product space  $\langle -1 \rangle$  and use the fact that  $v \cdot v$  is congruent mod 8 to 1 and -1, respectively, on  $\langle 1 \rangle$  and  $\langle -1 \rangle$ , we get that  $v \cdot v \equiv \sigma(V) \pmod{8}$ . This is called van der Blij's lemma, and gives an algebraic understanding of expression (2.4). For more arithmetic details, see Ref. [51].

**Remarks. 1.** *Constraints implied by the characteristic element.* When  $v$  is a characteristic element then it can be checked that the determinant and the rank of the bilinear form  $\varphi$  are constrained by

$$\text{rank } \varphi + \det \varphi \equiv \varphi(v, v) + 1 \pmod{4}. \quad (3.32)$$

In particular, if  $\varphi$  is unimodular then the rank of  $\varphi$  is given by the value mod 4 of that bilinear form at the characteristic element. More general statements will be given in section 4.2.

**2.** *When the characteristic can be set to zero.* For type II inner product spaces, we can safely set  $v = 0$ . In this case, the signature is divisible by 8, so that  $\frac{1}{8}\sigma(V)$  is an integer.

**Insight from the partition function.** In [56] [57] the partition function of the M5-brane was outlined. This was put on firm mathematical ground in [26]. Constructing the partition function uses the fact that on an 8-manifold  $M^8$  the expression

$$\frac{1}{8} \int_{M^8} (\lambda^2 - L(M^8)) \quad (3.33)$$

is an integer, where  $\lambda$  is the integral lift of the Wu class  $v_4$ . As we saw above, this has an algebraic explanation: the square of the norm of a characteristic element of a non-degenerate symmetric bilinear form over  $\mathbb{Z}$  is always congruent to the signature mod 8. For manifolds of dimension  $4k$ , the characteristic elements for the intersection pairing in the middle dimension are the integer lifts  $\lambda$  of the Wu class  $v_{2k}$ . The expression (3.33) is then an integer, and its variation under to  $\lambda \mapsto \lambda + 2x$  gives a quadratic refinement of the intersection pairing. There are a lot of structural similarities between the M5-brane and type IIB string theory. The Chern-Simons construction for the partition function type IIB string theory amounts to forming a circle bundle and then going to the bounding manifold and constructing the corresponding line bundle over the intermediate Jacobian. This construction for type IIB string theory requires the vanishing of the Spin cobordism group  $\Omega_{11}^{\text{Spin}}(K(\mathbb{Z}, 6))$  of the Eilenberg-MacLane space  $K(\mathbb{Z}, 6)$  representing the type IIB field in degree five, conjectured to be the case in [56]. Witten's conjecture is proved by Igor Kriz and the author in [30], thus allowing the applicability of the Hopkins-Singer construction to type IIB string theory. Indeed, the 12-dimensional version of expression (3.33) was assumed in [7] to describe the Chern-Simons action in type IIB string theory. This is also the basis of our discussion on the antisymmetric tensor field. Note that self-duality was not an issue in arriving at the construction for the M5-brane [56] [26], and hence we follow that line of thought for type IIB string theory.

For the Chern-Simons construction in type IIB, we need to consider the 6th Wu class  $v_6$  on the 12-dimensional extension. One might wonder what will happen to the 5th Wu class on  $X^{10}$  itself.

**The fifth Wu class.** Assuming that  $H_4(X^{10})$  has no 2-torsion (cf. Section 3.3), then the Wu class  $v_5$  vanishes if and only if there is a matrix representative for the intersection pairing so that all the diagonal entries are even. This happens if and only if every matrix representative for the intersection pairing has even diagonal entries. In fact, by the Wu formula, the odd degree class  $v_5$  is a composite class each of whose summands involves the first Stiefel-Whitney class  $w_1$ ; since we are dealing with oriented manifolds,  $v_5$  will always be zero in the situations we consider.

### 3.7 Wu Structure via Spin structures

The study of Wu structures can be done in a very general setting with minimal topological structure and without the need for any geometry. Consider the topological space  $BSO[v_6]$  over  $BSO$ , the classifying space for the stable orthogonal group, with fiber the Eilenberg-MacLane space  $K(\mathbb{Z}_2, 5)$

$$\begin{array}{ccc} K(\mathbb{Z}_2, 5) & \xrightarrow{=} & K(\mathbb{Z}_2, 5) \\ \downarrow & & \downarrow \\ BSO[v_6] & \xrightarrow{\quad} & EK(\mathbb{Z}_2, 6) \\ \downarrow \pi & & \downarrow \\ BSO & \xrightarrow{k} & K(\mathbb{Z}_2, 6) \end{array} \quad (3.34)$$

The  $k$ -invariant of this fibration is an element  $v_6$  in the cohomology  $H^6(BSO; \mathbb{Z}_2)$  defined by the 6th Wu class of the universal bundle  $\xi$  over  $BSO$ . A Wu structure on  $X^{10}$  means a lifting  $\tilde{\nu} : X^{10} \rightarrow BSO[v_6]$  of the classifying map  $\nu : X^{10} \rightarrow BSO$  from  $BSO$  to the connected cover  $BSO[v_6]$ , that is there is a diagram

$$\begin{array}{ccc} & & BSO[v_6] \\ & \nearrow \tilde{\nu} & \downarrow \pi \\ X^{10} & \xrightarrow{\nu} & BSO \end{array} \quad (3.35)$$

such that  $\pi \circ \tilde{\nu} = \nu$ .

Let  $\eta$  be a vector bundle over our 10-manifold  $X^{10}$  with vanishing first and second Stiefel-Whitney classes  $w_1(\eta) = w_2(\eta) = 0$ . Then by the Adem relations, the Wu class  $v_6(\eta)$  is always zero. Therefore, a Spin structure leads to a Wu structure. The situation is summarized in the following diagram

$$\begin{array}{ccccc} & & & & BSO[v_6] \\ & & & \nearrow \tilde{r} & \downarrow \pi \\ X^{10} & \xrightarrow{\quad} & BSpin & \xrightarrow{\quad} & BSO \\ & \searrow \nu & & & \end{array} \quad (3.36)$$

The possible lifts  $\tilde{r}$  are classified by  $H^5(BSpin; \mathbb{Z}_2)$ , which is zero. Therefore, there is a unique lift and hence each Spin structure uniquely determines a Wu structure.

Note that the Wu formula and Poincaré duality imply that the Wu class  $v_6$  will vanish on all Spin 10-manifolds. Similarly this holds in eleven dimensions.<sup>8</sup> However, this is generally not the

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<sup>8</sup>Later will consider relative Wu classes.

case in twelve dimensions. Note that one might naively expect that  $v_6$ , being a middle cohomology class in twelve dimensions, will vanish in analogy to  $v_5$  vanishing in ten dimensions (see end of Section 3.6). However, this is not the case; the main point is that there is a big difference in the structure of Wu classes in the even and odd degree cases. The appearance of  $v_6$  in eleven and twelve dimensions will be discussed towards the end of section 4.3; in fact there we will encounter a relative version of this class.

## 4 Diffeomorphisms

In this section we consider diffeomorphisms and their manifestation in ten, eleven, and twelve dimensions in detail, making use of the arguments and constructions in the previous sections. Consider a diffeomorphism  $f : X^{10} \rightarrow X^{10}$  which preserves some structure on the 10-manifold  $X^{10}$ . We certainly would like for  $f$  to preserve the orientation on  $X^{10}$ . In addition, we also would like to preserve further structure:

1. *The Spin structure:* We will consider Spin-preserving diffeomorphisms as well as the stronger notion of Spin-diffeomorphisms.
2. *The quadratic refinement:* We would like for the diffeomorphisms to leave invariant the quadratic form coming from the middle cohomology (as described in previous sections).

Preserving the first structure is natural since  $X^{10}$  is assumed to be a Spin manifold. The second structure is dictated by the fact that we are considering nontrivial middle cohomology involving such refinements. We will see that the above two types of diffeomorphisms are related, that is preserving a Spin structure is related to preserving the corresponding quadratic forms. In summary, we would like to study the action of the diffeomorphism group on

- (i) bilinear forms;
- (ii) quadratic refinements;
- (iii) middle cohomology.

**The mapping torus of a diffeomorphism** Let us temporarily abbreviate the 11-dimensional mapping torus  $(X^{10} \times S^1)_f$  by  $X_f$ . If  $f$  and  $g$  are diffeomorphisms of  $X^{10}$  then the cobordism class of the composition decomposes into classes in  $\Omega_{11}$ , the cobordism group of closed oriented differentiable 11-manifolds, as  $[X_{f \circ g}] = [X_f] + [X_g]$ .

### 4.1 Diffeomorphisms preserving the Spin structure

Consider a 10-manifold  $X^{10}$  with frame bundle  $F(X)$  and Spin bundle  $S(X)$  with structure groups  $\text{SO}(10)$  and  $\text{Spin}(10)$ , respectively. Given an orientation-preserving diffeomorphism  $f : X^{10} \rightarrow X^{10}$ , the differential  $df$  of  $f$  gives a diffeomorphism at the level of the frame bundle  $df : F(X) \rightarrow F(X)$ , and hence an isomorphism  $(df)^* : H^1(F(X); \mathbb{Z}_2) \rightarrow H^1(F(X); \mathbb{Z}_2)$ . Such a diffeomorphism  $f$  preserves the Spin structure  $\omega$  if  $(df)^*(\omega) = \omega$  in  $H^1(F(X); \mathbb{Z}_2)$ . This is also called a *Spin preserving diffeomorphism*. On the other hand, a Spin diffeomorphism  $\hat{f}$  of  $(X^{10}, \omega)$  is a pair  $\hat{f} = (f, b)$  consisting not only of a Spin preserving diffeomorphism  $f$  but also of a bundle map  $b : S(X) \rightarrow S(X)$

covering  $f$ ; then there is a commutative diagram

$$\begin{array}{ccc} S(X) & \xrightarrow{b} & S(X) \\ \downarrow & & \downarrow \\ F(X) & \xrightarrow{df} & F(X) \end{array} . \quad (4.1)$$

**Spin diffeomorphisms.** A *Spin diffeomorphism* is a quadruple  $(X^{10}, \omega, f, h)$  where [29] [31] [18]

- $w : X^{10} \rightarrow B\text{Spin}$  is a Spin structure.
- $f : X^{10} \rightarrow X^{10}$  is a diffeomorphism.
- $h : I \times X^{10} \rightarrow B\text{Spin}$  is a Spin structure on  $[0, 1] \times X^{10}$  such that  $h_0 = \omega$  and  $h_1 = \omega \circ f$ .

For a given diffeomorphism  $f$  with this property there are exactly two homotopy classes of choices for  $h$  since  $H^1(I \times X^{10}, \partial I \times X^{10}; \mathbb{Z}) \cong \mathbb{Z}_2$ . Because of the double covering map  $\text{Spin} \rightarrow \text{SO}$ , the following map is also two-to-one

$$\{\text{Spin diffeomorphisms}\} \xrightarrow{2:1} \{\text{Diffeomorphisms preserving Spin structures}\} . \quad (4.2)$$

Therefore, as far as Spin structures are concerned, we can have two quotients of the space of metrics  $\mathcal{M}_{\text{met}}$  on  $X^{10}$ , namely

$$\mathcal{M}_{\text{met}} / \{\text{Spin diffeomorphisms}\} \quad \text{and} \quad \mathcal{M}_{\text{met}} / \{\text{Diffeomorphisms preserving Spin structures}\} .$$

**The mapping torus of a Spin diffeomorphism.** One way of defining the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$  in this case is to take (cf. [18])  $(Y^{11}, \omega)$  to be the Spin manifold formed as follows:  $\mathbb{Z}$  acts on  $\mathbb{R} \times X^{10}$  by  $(n, (r, x)) \mapsto (r - n, f^n(x))$  and then  $Y^{11} := \mathbb{R} \times_{\mathbb{Z}} X^{10}$ , with the Spin structure  $\omega_h$  induced by  $h$ .

**Example 8.** Consider  $X^{10} = S^5 \times S^5$  with the Spin structure given by the stable trivialization of the tangent bundle  $TX^{10} \oplus \mathcal{O}^2 = (TS^5 \oplus \mathcal{O}) \times (TS^5 \oplus \mathcal{O}) = (\mathcal{O}^6) \times (\mathcal{O}^6)$ . As in [31], consider the mapping torus  $(X^{10} \times S^1)_f = S^5 \times S^5 \times S^1$  associated to the identity diffeomorphism  $(\text{id}, \text{id})$  on the two factors in  $X^{10}$ , and let  $\Delta : S^5 \hookrightarrow S^5 \times S^5$  be the diagonal map  $\Delta(x) = (x, x)$ . The normal bundle of  $\Delta(S^5) \times \{\text{pt}\}$  in  $S^5 \times S^5 \times S^1$  has a natural trivialization  $TS^5 \oplus \mathcal{O} = \mathcal{O}^6$ . The triviality of the normal bundle allows us to use surgery to attach the handle made up of the product of two 6-disks  $\mathbb{D}^6 \times \mathbb{D}^6$  to  $S^5 \times S^5 \times S^1 \times [0, 1]$  by gluing  $S^5 \times \mathbb{D}^6$  to a neighborhood of  $\Delta(S^5) \times \{\text{pt}\} \times \{1\}$  in  $S^5 \times S^5 \times S^1 \times \{1\}$  via the trivialization. In the resulting manifold  $W$ , we have embedded the 6-disk  $\mathbb{D}^6$  with trivial normal bundle and boundary the diagonal 5-sphere  $\Delta(S^5)$ . Let  $p$  be a base point of  $S^5$ . Then the quadratic form corresponding to the diagonal map is

$$\begin{aligned} q_X(\Delta_*(S^5)) &= q_X([S^5 \times p] + [p \times S^5]) \\ &= q_X([S^5 \times p]) + q_X([p \times S^5]) + \phi_X([S^5 \times p], [p \times S^5]) . \end{aligned}$$

The two quadratic forms on the right hand side are equal as we can exchange the two factors by a Spin preserving diffeomorphism. This implies that the left hand side is equal to the intersection form which is odd, that is  $q_X(\Delta_*(S^5)) = 1 \pmod{2}$ .

## 4.2 Diffeomorphisms preserving the quadratic structure

We would like to (also) preserve the quadratic form, as we mentioned above. Ultimately, what we need is to quotient the space of Riemannian metrics by an intersection of diffeomorphisms preserving the Spin structure (or Spin diffeomorphisms) with diffeomorphisms preserving quadratic refinement. One way to ensure we get the latter is to have the diffeomorphism induce an isometry on the quadratic forms.

**Isometric quadratic forms.** Let  $q_1$  and  $q_2$  be two quadratic forms. An *isometry*  $f : q_1 \rightarrow q_2$  is a linear map between the underlying vector spaces  $V_{q_1} \rightarrow V_{q_2}$  such that  $q_1(x) = q_2(f(x))$  for all  $x \in V_{q_1}$ . If such an isometry exists, we write  $q_1 \simeq q_2$  and say  $q_1$  and  $q_2$  are isometric.

**Preserving the quadratic refinement.** We have seen in section 3.5.1 that the  $\mathbb{Z}$ -intersection pairing leads to a corresponding  $\mathbb{Z}_2$ -intersection pairing, which can be identified with a map  $q : H_5(X^{10}; \mathbb{Z}) \rightarrow \mathbb{Z}_2$  satisfying relation (3.27). If  $f : X^{10} \rightarrow X^{10}$  is a Spin preserving diffeomorphism of  $(X^{10}, \omega)$ , then by naturality of the construction in Ref. [31] (of which we will make more use in section 4.3), we have  $q(f_*(x)) = q(x)$  for all  $x$  in  $H_5(X^{10}; \mathbb{Z})$ . Therefore, the diffeomorphism  $f$  preserves the quadratic refinement.

**Preserving quadratic forms.** There are various invariants that are defined to determine whether quadratic forms over an arbitrary field  $\mathbb{F}$  are isometric. These invariants live in Galois cohomology  $H^i \mathbb{F}$  corresponding to the field  $\mathbb{F}$ . The following invariants correspond to cohomology classes of ascending degrees, starting from degree 0. They are all defined on the Witt group  $W\mathbb{F}$  of the field  $\mathbb{F}$ . In addition, they behave like obstructions in the sense that the  $j$ -th invariant is a homomorphism when restricted to the kernel of the  $(j-1)$ -th invariant. These invariants  $\text{Inv}_j(q)$  are:

1. *Dimension:* In order to get an invariant that vanishes on hyperbolic forms, one considers

$$\text{Inv}_0(q) = \dim q \pmod{2} \in \mathbb{Z}_2 = H^0 \mathbb{F}. \quad (4.3)$$

2. *Discriminant:* For  $q$  a quadratic form of dimension  $n$ ,

$$\text{Inv}_1(q) = (-1)^{n(n-1)/2} \det q \in \mathbb{F}^\times / \mathbb{F}^{\times 2} = H^1 \mathbb{F}. \quad (4.4)$$

3. *Clifford invariant:* This is an invariant of the Clifford algebra or the even Clifford algebra, depending on the dimension invariant, and takes values in the 2-exponent part of the Brauer group of  $\mathbb{F}$

$$\text{Inv}_2(q) = \begin{cases} [C\ell(q)] \in {}_2\text{Br}(\mathbb{F}) & \text{if } \dim q \text{ is even;} \\ [C\ell_0(q)] \in {}_2\text{Br}(\mathbb{F}) & \text{if } \dim q \text{ is odd.} \end{cases} \quad (4.5)$$

In general there are more invariants,  $\text{Inv}_n : \ker \text{Inv}_{n-1} \rightarrow H^n \mathbb{F}$  for all  $n \geq 0$ ; however not all are needed due to a truncation process. Then the problem of deciding whether two quadratic forms  $q_1, q_2$  over  $\mathbb{F}$  are isometric can be solved by computing cohomology classes. First, one checks that  $\dim q_1 = \dim q_2$ . If this holds then one checks that  $q_1 - q_2$  is hyperbolic. This process can be tested by successively ensuring that  $\text{Inv}_i(q_1 - q_2) = 0$  for  $i$  running over the ordered set  $\{0, 1, \dots, d\}$ , where the process truncates at  $i = d$  for  $2^d \leq \dim q_1 + \dim q_2$  via the so-called Arason-Pfister Hauptsatz (see [53]). Note that for  $\mathbb{F} = \mathbb{Z}_2$ , the dimension and the Arf invariant form a complete invariant.

We have seen in relation (3.32) how (essentially) the sum of the first two invariants above– but for the bilinear form– is constrained by the value of the bilinear form at a characteristic element.

We will need the following related concept.

**Isometric structure.** An isometric structure over  $R = \mathbb{Z}$  or  $\mathbb{Z}_2$  is a triple  $(V, s, \mathcal{I})$ , where

- $V$  is a free finite-dimensional  $R$ -module.
- $s : V \times V \rightarrow R$  an antisymmetric unimodular bilinear form.
- $\mathcal{I} : V \rightarrow V$  is an isometry of  $(V, s)$  into itself, i.e. for all  $x, y$  in  $V$ ,  $s(x, y) = s(\mathcal{I}(x), \mathcal{I}(y))$ .

For us  $V$  is the middle cohomology,  $s$  is the intersection pairing, and  $h$  is the isometry of the intersection pairing (later this will be induced from a diffeomorphism  $f$  as  $f_*$  on the homology). The sum of two isometric structures is defined by the orthogonal direct sum  $(V_1, s_1, \mathcal{I}_1) + (V_2, s_2, \mathcal{I}_2) = (V_1 \oplus V_2, s_1 \oplus s_2, \mathcal{I}_1 \oplus \mathcal{I}_2)$ . The abelian group of equivalence classes  $[V, s, \mathcal{I}]$  of isometric structures denoted by  ${}^9 W_{-1}(\mathbb{Z}; R)$ , the *Witt group* of antisymmetric structures over  $R$ . For  $R$  equal to  $\mathbb{Z}$  or  $\mathbb{Q}$ , the Witt group is infinite-dimensional and is given by  $W_{-1}(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ . The torsion-free part is detected by the equivariant signature and the torsion is related to number theoretic invariants that we will not consider here (see [38]).

Now for  $(X^{10}, f)$  a diffeomorphism of a 10-dimensional closed manifold, the intersection form  $s$  on  $H_5(X^{10}; \mathbb{Z})/\text{Tor}$  is antisymmetric and unimodular by Poincaré duality. The diffeomorphism  $f$  induces an isometry  $f_* : H_5(X^{10}; \mathbb{Z})/\text{Tor} \rightarrow H_5(X^{10}; \mathbb{Z})/\text{Tor}$ . The isometric structure  $I(X^{10}, f)$  of a diffeomorphism  $(X^{10}, f)$  is defined as

$$[H_5(X^{10}; \mathbb{Z})/\text{Tor}, \phi_X^*, f_*] \in W_{-1}(\mathbb{Z}; \mathbb{Z}) . \quad (4.6)$$

This equivalence class in the Witt group is a cobordism invariant. See [29] for more details.

**Preserving Spin structure vs. preserving quadratic structure.** As mentioned at the beginning of Section 4 and the introduction to the current Section, one way to ensure preserving both the Spin structure and the quadratic structure is to restrict to those diffeomorphisms which lie in the intersection of the diffeomorphisms preserving the first and those preserving the second. There is in fact a map from the set of Spin structures on  $X^{4k+2}$  to the set of quadratic refinements of the mod 2 intersection pairing on  $H_{2k+1}(X^{4k+2}; \mathbb{Z}_2)$  [12]. The set of Spin structures is  $H^1(X; \mathbb{Z}_2)$  and the set of quadratic refinements is the 2-exponent group  ${}_2H^{2k+1}(X^{10}, U(1))$  (cf. [7]). However, this map is neither injective nor surjective in general, so that knowing one side of the map does not in general tell us about the other in any complete way. However, in the case of Riemann surfaces, corresponding to  $k = 0$ , the map is an isomorphism. What we can do is assume that one of the sets is a subset of the other set. For example, we can assume an injection  $H^1(X^{10}; \mathbb{Z}_2) \hookrightarrow {}_2H^5(X^{10}, U(1))$ , so that preserving the quadratic refinement also preserves the Spin structure. Depending on whether the number of Spin structures is large, we can also assume an injection the other way. At any rate, as mentioned in the remarks at the end of section 2.4, we do not need to go into such specifications in order to arrive at the conclusions on anomaly cancellation.

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<sup>9</sup>The -1 subscript refers to antisymmetric.

### 4.3 Diffeomorphism on (almost) middle cohomology in 11 and 12 dimensions

We have seen in Section 3.5.2 that the torsion subgroup in eleven dimensions and the corresponding linking pairing, equation (3.28), are related to the Arf invariant. In this section we will see how both data, the torsion subgroup  $T_5((X^{10} \times S^1)_f)$  and the linking pairing  $L$ , can be described in terms of the induced mapping  $f_* : H_5(X^{10}; \mathbb{Z}) \rightarrow H_5(X^{10}; \mathbb{Z})$  on the middle-dimensional cohomology of the base 10-manifold. We will make use of the construction in Ref. [31].

**Extension to the mapping torus.** Consider an element  $y \in H_5(X^{10}; \mathbb{Z})$ . We would like to see how much  $y$  changes under the action of  $f_*$ , that is to the new element  $f_*y$ . To that end, consider the difference  $y - f_*y$ , represented by the action of the map  $(1 - f_*)$  on the element  $y$ . Requiring this difference to be zero might be too much to ask as then we would be saying that these element are actually invariant. However, we would like to do something close, namely consider the above difference to be a nonzero multiple of a nontrivial element  $x$  in  $H_5(X^{10}; \mathbb{Z})$ . Hence we consider a summand in  $H_5(X^{10}; \mathbb{Z})$  given by

$$\mathcal{A} = \{x \in H_5(X^{10}; \mathbb{Z}) \mid Nx = y - f_*y \text{ for some nonzero integer } N \text{ and some } y \in H_5(X^{10}; \mathbb{Z})\} . \quad (4.7)$$

On this group, define the rational bilinear pairing  $\mathcal{B} : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{Q}$  by the formula

$$\mathcal{B}(x_1, x_2) = \frac{1}{N} \cdot \phi_X^*(x_1, x_2) , \quad (4.8)$$

where  $Nx_2 = y_2 - f_*(y_2)$  and  $\phi_X^*$  is the intersection pairing on  $X^{10}$ . The image of the map  $(1 - f_*) : H_5(X^{10}; \mathbb{Z}) \rightarrow H_5(X^{10}; \mathbb{Z})$  is contained in  $\mathcal{A}$  and the quotient  $[\mathcal{A}/\text{im}(1 - f_*)]$  is a finite abelian group, i.e. a lattice. In fact, the inclusion  $\iota : X^{10} \hookrightarrow (X^{10} \times S^1)_f$  of  $X^{10}$  into  $X^{10} \times 0$  leads to an isomorphism of torsion groups  $[\mathcal{A}/\text{im}(1 - f_*)] \cong T_5((X^{10} \times S^1)_f)$ . Indeed, consider homology long exact sequence

$$\longrightarrow H_5(X^{10}; \mathbb{Z}) \xrightarrow{(1-f_*)} H_5(X^{10}; \mathbb{Z}) \xrightarrow{i_*} H_5(Y^{11}; \mathbb{Z}) \longrightarrow H_4(X^{10}; \mathbb{Z}) \xrightarrow{(1-f_*)} H_4(X^{10}; \mathbb{Z}) . \quad (4.9)$$

Since  $H_5(X^{10}; \mathbb{Z})$  is assumed to be torsion-free, then so is  $H_4(X^{10}; \mathbb{Z})$  by Poincaré duality.<sup>10</sup> This implies that the torsion subgroup  $T_5(Y^{11}) \subset H_5(Y^{11}; \mathbb{Z})$  does not get any contribution from elements in  $\ker(1 - f_*) : H_4(X^{10}; \mathbb{Z}) \rightarrow H_4(X^{10}; \mathbb{Z})$ . This gives the desired result.

Now the bilinear form  $\mathcal{B}$  on the set  $\mathcal{A}$  induces a corresponding bilinear form  $\mathcal{B}'$  on this torsion group

$$\mathcal{B}' : [\mathcal{A}/\text{im}(1 - f_*)] \times [\mathcal{A}/\text{im}(1 - f_*)] \rightarrow \mathbb{Q}/\mathbb{Z} , \quad (4.10)$$

which is exactly the linking pairing  $L$  on  $T_5(Y^{11})$ ; cf. equation (3.28).

Consider a quadratic refinement  $q$  which is compatible with  $f$  in the sense that  $q(f_*(x)) = q(x)$  for all  $x$  in  $H_5(X^{10}; \mathbb{Z})$  (cf. Section 4.2). Associated to this quadratic refinement there is a mapping  $\tilde{Q}[q] : \mathcal{A} \rightarrow \mathbb{Q}/\mathbb{Z}$ , defined by  $\tilde{Q}[q](x) = \frac{1}{2}\mathcal{B}(x, x) + j(q(x))$ , where  $j : \mathbb{Z}_2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$  represents the inclusion  $j(1) = \frac{1}{2}$ ,  $j(0) = 0$ . From the general construction of Ref. [31], the mapping  $\tilde{Q}$  induces a quadratic refinement

$$Q[q] : [\mathcal{A}/\text{im}(1 - f_*)] \rightarrow \mathbb{Q}/\mathbb{Z} \quad (4.11)$$

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<sup>10</sup>See Section 3.3.



of the nonsingular pairing  $\mathcal{B}'$  in (4.10) (which coincides with the linking pairing  $L$  on the mapping torus given in (3.28)). Recall that this quadratic refinement was defined solely from the the map  $f_*$  and the quadratic form  $q$ , both on the basic middle homology  $H_5(X^{10}; \mathbb{Z})$ .

**Extension to twelve dimensions.** Consider  $Y^{11} = (X^{10} \times S^1)_f$  as the boundary of a compact smooth oriented 12-manifold  $Z^{12}$ . We need to choose a *relative Wu class*  $v' \in H^6(Z^{12}, Y^{11}; \mathbb{Z}_2)$  whose restriction  $v'|_{Z^{12}}$  is the 6th Wu class  $v_6(Z^{12}) \in H^6(Z^{12}; \mathbb{Z}_2)$ , and which is compatible with the quadratic refinement  $Q$ . The most convenient choice is  $v' = 0$ .<sup>11</sup> But then for all relative homology classes  $b \in H_6(Z^{12}, Y^{11}; \mathbb{Z})$  with  $\partial b$  a torsion class in  $H_5(Y^{11}; \mathbb{Z})$  we should have

$$Q[q](\partial b) = -\frac{1}{2}\phi_Y(b, \tilde{b}) \pmod{\mathbb{Z}} \text{ in } \mathbb{Q}/\mathbb{Z}, \quad (4.12)$$

where  $\tilde{b}$  is some choice of rational class in  $H_6(Z^{12}; \mathbb{Q})$  which has the same image as  $b$  in the relative rational homology group  $H_6(Z^{12}, Y^{11}; \mathbb{Q})$ . This can be checked explicitly using chains [31].

#### 4.4 Description via the Rochlin invariant

In this section we will see how the expression for the anomaly in type IIB involving the combination of eta invariants on the mapping torus (cf. expressions (2.8) and (2.22)) is encoded in the Rochlin invariant. This will be an overview and an application of the mathematical results in [34] [31] [18].

On the Spin manifold  $Y^{11} = (X^{10} \times S^1)_f$  there exists a well-defined  $\mathbb{Z}_{16}$ -invariant  $R(Y^{11})$  given by the formula [34]

$$R(Y^{11}) = \sigma(Z^{12}) \pmod{16} \in \mathbb{Z}_{16}. \quad (4.13)$$

This Rochlin invariant is well-defined; this follows from the Novikov additivity of the signature and the divisibility in the closed case, i.e. Ochanine's result [41] that the signature  $\sigma(Z^{12})$  on the intersection pairing  $\phi_Z^* : H_6(Z^{12}; \mathbb{Z}) \otimes H_6(Z^{12}; \mathbb{Z}) \rightarrow \mathbb{Z}$  of the middle-dimensional homology of a compact *closed* Spin manifold 12-manifold  $Z^{12}$  is divisible by 16.<sup>12</sup> The value of the Rochlin invariant modulo 8 is independent of the choice of Spin diffeomorphism  $F = (f, b)$  covering  $f$  and only depends on data related to the middle (co)homology, namely:

1. The quadratic mapping  $q_\omega : H_5(X^{10}; \mathbb{Z}) \rightarrow \mathbb{Z}_2$  defined by the Spin structure  $\omega$ . This is a quadratic refinement of the intersection pairing on  $X^{10}$  constructed by Brown [12].
2. The induced map  $f_* : H_5(X^{10}; \mathbb{Z}) \rightarrow H_5(X^{10}; \mathbb{Z})$  on the middle-dimensional homology of  $X^{10}$ .

Since  $Y^{11}$  with its Spin structure  $\omega$  always bounds, then from [31], the Rochlin invariant is given by

$$R(Y^{11}, \omega) = \frac{1}{2}\eta(Y^{11}, \mathcal{S}) + 4[h(Y^{11}; D_{TY}) + \eta(Y^{11}; D_{TY})] - 16\eta(Y^{11}; D) \pmod{16}. \quad (4.14)$$

We now consider the Rochlin invariant in the presence of some structure. From [13], we have for the relative cohomology  $H^6(BSO, BSO[v_6]; \mathbb{Z}_2) \cong \mathbb{Z}_2$  and so this group contains a unique nonzero element  $v$ . Let  $g : Z^{12} \rightarrow BSO$  denote the classifying map of the stable normal bundle of  $Z^{12}$ . The

<sup>11</sup>See the end of Section 3.6 as well as Section 3.7 for a discussion on Wu classes.

<sup>12</sup>So had  $Z^{12}$  been closed then showing absence of global anomaly would have been straightforward.

pair of mappings  $(g, \tilde{v}) : (Z^{12}, X^{10}) \rightarrow (BSO, BSO[v_6])$  can be used to pullback  $v$  to a cohomology class  $\tilde{v}$  in the relative cohomology group  $H^6(Z^{12}, X^{10}; \mathbb{Z}_2)$ ; indeed, we know that  $X^{10}$  admits a Wu-structure (see end of Section 4.3).

From the point of view of the mapping torus  $Y^{11} = (X^{10} \times S^1)_f$ , we need to consider the corresponding pairing  $L$  on the torsion subgroup  $T^5(Y^{11})$  as well as the quadratic refinement  $Q_L$  (see expression (3.30)). Using the quadratic refinement  $Q_L$ , one can assign to such a class  $\tilde{v}$  a modulo 8 invariant  $\tilde{v}_Q^2$  such that the following relation holds [34]

$$\tilde{v}_Q^2 - A(Y^{11}, Q_L) = \sigma(Z^{12}) \pmod{8}. \quad (4.15)$$

If  $Z^{12}$  is taken to be a Spin manifold then the maps to  $BSO$  and  $BSO[v_6]$  factor through  $B\text{Spin}$ , so that the pair  $(g, \tilde{v})$  induces a trivial map between relative cohomology groups (see diagram (3.35)) and in this case we have  $\tilde{v} = 0$  and  $\tilde{v}_Q = 0$ .

Let us consider  $X^{10}$  to be Spin with a Spin structure  $\omega$ . From Ref. [12], the Spin structure  $\omega$  gives a canonical refinement  $q$  of the  $\mathbb{Z}_2$ -intersection pairing, that is

$$\begin{aligned} q : H_5(X^{10}; \mathbb{Z}) &\longrightarrow \mathbb{Z}_2 \\ q(x + y) - q(x) - q(y) &= \phi_X^*(x, y) \pmod{2}. \end{aligned}$$

Then in this case where all manifolds are Spin, and using [31], we have that the Rochlin invariant, the Arf invariant, and the signature are related as

$$R(Y^{11}) = \sigma(Z^{12}) = -A(Y^{11}, Q_L) \pmod{8}. \quad (4.16)$$

**The Rochlin invariant in terms of the Arf invariant.** We have seen towards the end of Section 4.3 that a quadratic refinement can be constructed on the mapping torus starting from the action of the diffeomorphism on the middle cohomology of the base 10-manifold  $X^{10}$ , via  $f_*$ , and from the corresponding quadratic form  $q$ . This quadratic refinement satisfies some compatibility conditions spelled there (cf. equation (4.12)). Thus with compatibility, via [13], the Rochlin invariant of the mapping torus is given by the Arf invariant of this quadratic form  $Q[q]$  in (4.11), that is

$$R((X^{10} \times S^1)_f, \omega') = -A(Q[q]) \pmod{8}. \quad (4.17)$$

**Example 9: Products with  $b_5 = 0$ .** We can consider the case of product manifolds with the possibility that diffeomorphisms on one or more of the factors are trivial. Take  $X^{10}$  to be the product manifold  $T^2 \times \mathbb{H}P^2$  of a two-torus with the quaternionic projective plane. Take  $\alpha := (T^2, \omega, f, h) - (T^2, \omega, \text{id}, h')$  where  $f$  is given by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\omega$  is the standard Spin structure of  $T^2 = \mathbb{R}^2/\mathbb{Z}^2$  (with nontrivial Arf invariant),  $h$  an appropriate homotopy and  $h'$  a constant homotopy. Then the Rochlin invariant is even,  $R(\alpha) \equiv 0 \pmod{2}$ . Now consider the quaternionic projective plane  $\mathbb{H}P^2$  and take  $\beta := (\mathbb{H}P^2, \omega, \text{id}, h)$  with any Spin structure, identity diffeomorphism and constant homotopy. Then the product  $(T^2 \times \mathbb{H}P^2)_{f \times \text{id}} = (T^2 \times S^1)_f \times \mathbb{H}P^2$  is a Spin boundary of  $M^4 \times \mathbb{H}P^2$  if  $\partial M^4 = (T^2 \times S^1)_f$ . Since the signature is multiplicative, the Rochlin invariant of the product is [18]  $R(\alpha \times \beta) = \text{sign}(M^4 \times \mathbb{H}P^2) = \text{sign}(M^4) \cdot \text{sign}(\mathbb{H}P^2)$ , which is equal to  $R(\alpha)$  since the signature of  $\mathbb{H}P^2$  is 1. We will consider this example further in section 4.5.

Note that this allows us to make use of the transparent Riemann surface case, for which there is a one-to-one correspondence between the set of Spin structures and the set of quadratic refinements. The general case is reduced to this particular case by taking  $X^{10} = \Sigma_g \times \mathbb{R}^8$ , where  $\Sigma_g$  is a Riemann surface of genus  $g$ .

**Variation of Spin structure on  $X^{10}$  and the Arf invariant.** Now we consider the situation where  $X^{10}$  has (at least) two Spin structures. This means that  $X^{10}$  has to satisfy  $|H^1(X^{10}; \mathbb{Z}_2)| \geq 2$ . Let  $f : X^{10} \rightarrow X^{10}$  be an orientation preserving diffeomorphism which preserves two Spin structures  $\omega_1, \omega_2$  on  $X^{10}$ . Lift  $f$  to two Spin diffeomorphisms  $F_1 = (f, b_1)$  and  $F_2 = (f, b_2)$  which preserve the Spin structures  $\omega_1$  and  $\omega_2$ , respectively. Let  $\omega'_1, \omega'_2$  be the Spin structures on  $(X^{10} \times S^1)_f$  corresponding to these two choices  $F_1$  and  $F_2$ . In this situation, the Rochlin invariant of the difference  $R[(Y^{11}, \omega'_1) - (Y^{11}, \omega'_2)]$  is always defined since  $Y^{11}$  is always the Spin boundary of some 12-manifold  $Z^{12}$ .

Now let  $q_1, q_2 : H_5(X^{10}; \mathbb{Z}) \rightarrow \mathbb{Z}_2$  be the quadratic refinements of the intersection pairing determined by the Spin structures  $\omega'_1, \omega'_2$ , respectively. We need to look at the value of the difference  $q_2(y) - q_1(y)$  inside  $\mathbb{Q}/\mathbb{Z}$  via the embedding  $j : \mathbb{Z}_2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$ , for any element  $y$  in the group  $\mathcal{A}$ , defined in (4.7). In fact, there is a unique element  $z$  in the quotient  $[\mathcal{A}/\text{im}(1 - f_*)]$  such that  $j[q_2(y) - q_1(y)]$  coincides with the bilinear form  $\mathcal{B}'(y, z)$ , defined in (4.10), for all  $y$  in  $\mathcal{A}$ . Note that  $\mathcal{B}'$  is nonsingular, which is compatible with  $j(0) = 0$  and the fact that we take  $q_1$  and  $q_2$  to be distinct. Then, building on [31], the Rochlin invariant of the difference is essentially given by the difference of the Arf invariants of the corresponding quadratic forms

$$R[(Y^{11}, \omega'_1) - (Y^{11}, \omega'_2)] = A[Q(q_1)] - A[Q(q_2)] \pmod{8}. \quad (4.18)$$

**Variation of Spin structure and the Ochanine invariant.** Let  $(X^{10}, f)$  be a fixed connected Spin manifold and  $f$  a Spin diffeomorphism. Then there are exactly two homotopy classes of homotopies from  $f \circ \omega$  to  $\omega$ . In particular, there are exactly two Spin structures on the mapping torus corresponding to the identity diffeomorphism  $(X^{10} \times S^1)_{\text{id}} = X^{10} \times S^1$ , and let  $\bar{h}$  be the one nontrivial on  $S^1$ . Then  $X^{10} \times S^1$  is also a Spin boundary with respect to  $\bar{h}$ . The Ochanine invariant [41] is defined in our setting as

$$O(X^{10}, \omega) := R(X^{10}, \omega, \text{id}, \bar{h}) \in \mathbb{Z}_{16}. \quad (4.19)$$

Note that  $O(X^{10}, \omega) \in 8 \cdot \mathbb{Z}_{16} \cong \mathbb{Z}_2$  since  $2O(X^{10}, \omega) = R(2(X^{10}, \omega, \text{id}, \bar{h})) = 0$ . This invariant is always divisible by 8. Next, let  $h$  and  $h'$  be representatives of the two homotopy classes of homotopies joining  $\omega$  to  $\omega \circ f$ . Then, using [18], we have that the variation of the Rochlin invariant is given by the Ochanine invariant of the base

$$R(X^{10}, \omega, f, h) - R(X^{10}, \omega, f, h') = O(X^{10}, \omega). \quad (4.20)$$

In fact, as can be deduced from Ref. [14], both of the above variations are zero mod 8. Applications of this invariant to the partition function in M-theory is given in Ref. [45].

**Effect of torsion in middle (co)homology.** To which extent is the Rochlin invariant  $R$  determined by the induced map on  $H_5(X^{10}; \mathbb{Z})$ ? This will depend on whether or not torsion is present. The formulation in [31] gives a formula for  $R \pmod{8}$  in terms of the induced map, if  $H_5(X^{10}; \mathbb{Z})$

is torsion-free. As argued in [18] (in more generality than what we need) this condition cannot be dropped so that there cannot be a formula depending only on the induced map  $f_*$  on  $H_5(X^{10}; \mathbb{Z})$  and the Spin structure. An example which illustrates this is given towards the end of Section 4.5. There is a relative cohomology class  $x \in H^6(Z^{12}, Y^{11}; \mathbb{Z}_2)$  such that

$$\langle x \cup x, [Z^{12}, Y^{11}] \rangle + \dim_{\mathbb{Z}_2}(T^6(Y^{11}) \otimes \mathbb{Z}_2) \equiv \sigma(Z^{12}) \pmod{2}. \quad (4.21)$$

By Poincaré duality,  $T^6(Y^{11}) \cong T_5(Y^{11})$ . Furthermore, if we set  $x = 0$ , then the Rochlin invariant in this case can be calculated only mod 2 as

$$R(X^{10}, w) \equiv \dim_{\mathbb{Z}_2}(T_5(X^{10}) \otimes \mathbb{Z}_2) \pmod{2}. \quad (4.22)$$

#### 4.5 Description via cobordism invariants related to the Rochlin invariant

We have seen that the global anomaly formula involves the division of this linear combination of eta invariants by 8. It is then natural to ask whether this division leads to an integer or just a rational number. This makes a direct use of the results in [18] as well as the constructions in [29]. One of the byproducts is an explanation of the extension from the circle to the Riemann surface in diagram (2.5) and the discussion around it.

**Cobordism of diffeomorphisms.** The cobordism group of  $m$ -dimensional diffeomorphisms  $\Delta_m$  is the cobordism group of differentiable fiber bundles over  $S^1$  with  $(m+1)$ -dimensional total space and is given by the mapping torus. In the case of  $X^{10}$  in type IIB, we have to consider  $\Delta_{10}$ , which is not finitely generated nor finite-dimensional (even rationally). It is natural to ask how the cobordism group of 10-dimensional diffeomorphisms  $\Delta_{10}$  is related to other ‘more common’ cobordism groups. To answer this question, we would like to describe three homomorphisms from  $\Delta_{10}$ .

1. There is an obvious homomorphism from the cobordism group of diffeomorphisms  $\Delta_{10}$  to the cobordism group  $\Omega_{10} \cong \mathbb{Z}_2$  of closed oriented 10-manifolds given by forgetting the diffeomorphism and considering only the cobordism class of the underlying 10-manifold, that is  $[X^{10}, f] \mapsto [X^{10}]$ .
2. The mapping torus construction raises the dimension by one, and there is a homomorphism from  $\Delta_{10}$  to  $\Omega_{11} \cong \mathbb{Z}_2$ , the cobordism group of closed oriented 11-manifolds, given by  $[X^{10}, f] \mapsto [(X^{10} \times S^1)_f]$ . The image of this map, denoted  $\widehat{\Omega}_{11}$ , coincides with the kernel of the Hirzebruch signature operator  $\tau$  because the total space of a fibration over the circle has a vanishing signature [39].
3. The isometric structure (of section 4.2) leads to the surjective homomorphism  $I(X^{10}, f) : \Delta_{10} \rightarrow W_{-1}(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ . This third homomorphism is much more involved than the first two and requires the use of the Neumann invariant (see below).

Putting the three homomorphisms together, the ‘total’ homomorphism  $\Delta_{10} \rightarrow W_{-1}(\mathbb{Z}; \mathbb{Z}) \oplus \Omega_{10} \oplus \Omega_{11}$ , mapping  $[X^{10}, f]$  to  $(I(X^{10}, f), [X^{10}], [(X^{10} \times S^1)_f])$ , is an isomorphism [29]. Therefore, the cobordism group of diffeomorphisms of oriented 10-manifolds  $X^{10}$  is  $\Delta_{10} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .

**The case of  $X^{10}$  with extra structure.** The above discussion was for general manifolds with no special structure, i.e. for  $X^{10}$  only oriented. We will mainly be interested in the Spin case,

that is adding a Spin structure to the above discussion. However, we could also add other relevant structures, either more refined such as a String structure, or more crude such as a framing. Generally, if  $\mathbb{B}$  is e.g. BSpin, BString, BU, B1, corresponding to Spin structure, String structure, almost complex structure, and framing, respectively, then [29] the kernel of the homomorphism

$$\Delta_{10}^{\mathbb{B}} \longrightarrow W_-(\mathbb{Z}; \mathbb{Z}) \oplus \Omega_{10}^{\mathbb{B}} \oplus \Omega_{11}^{\mathbb{B}} \quad (4.23)$$

is a subgroup of  $\mathbb{Z}/\tau(\mathbb{B}, 12)\mathbb{Z}$ , where the denominator is the smallest positive signature of a closed 12-dimensional  $\mathbb{B}$ -manifold. We can consider the following cases:

1.  $X^{10}$  is *Spin*: If the 10-manifold  $X^{10}$  is Spin then we have to consider the cobordism group  $\Delta_{10}^{\text{Spin}}$  of 10-dimensional Spin diffeomorphisms. Here the relevant cobordism groups are in dimension eleven  $\Omega_{11}^{\text{Spin}} = 0$ , and dimension ten  $\Omega_{10}^{\text{Spin}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .
2.  $X^{10}$  is *String*: Here the relevant cobordism groups are  $\Omega_{10}^{\text{String}} = \mathbb{Z}_2 \oplus \mathbb{Z}_3$  and  $\Omega_{11}^{\text{String}} = 0$ .
3.  $X^{10}$  is *framed*: Many examples that we considered, including  $S^5 \times S^5$  are in fact framed manifolds.

We have, however, concentrated mostly on the Spin case in this paper.

**The Neumann invariant.** Again, let  $Y^{11} = \partial Z$  be the mapping torus  $(X^{10} \times S^1)_f$  for  $X^{10}$ . Then the Neumann invariant  $\mathcal{N}(X^{10}, f)$  is defined to be the signature of the symmetric bilinear form on  $H_5(X^{10}; \mathbb{Q})$  given by

$$(x, y) \longmapsto \phi_X((f_* - f_*^{-1})x, y) . \quad (4.24)$$

i.e.  $\sigma(Z^{12}) = \mathcal{N}(X^{10}, f)$ . Unlike the isometric structure described in Section 4.2 above, the Neumann invariant is not a cobordism invariant [40]. For example, let  $g : S^5 \times S^5 \rightarrow S^5 \times S^5$  be the clutching function of the sphere bundle of the tangent bundle of the 6-sphere  $S^6$ . Then with respect to the standard basis of  $H_5(S^5 \times S^5)$ ,  $g_*$  has the matrix description  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$  and the intersection form  $s$  has the matrix form  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . This gives the value of the Neumann invariant for the mapping torus  $\mathcal{N}(S^5 \times S^5, g) = 1$ . However, the mapping torus  $(S^5 \times S^5, g)$  is null-bordant since the sphere bundle bounds the disk bundle. Therefore,  $\mathcal{N}$  is not a cobordism invariant.

**Example 10.** Take  $X^{10}$  to be the product manifold  $T^2 \times \mathbb{H}P^2$ . Take  $\alpha := (T^2, \omega, f, h) - (T^2, \omega, \text{id}, h')$  where  $f$  is given by the matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\omega$  is the standard Spin structure of  $T^2 = \mathbb{R}^2/\mathbb{Z}^2$  (with nontrivial Arf invariant),  $h$  an appropriate homotopy, and  $h'$  a constant homotopy. Then the Neumann invariant is odd  $\mathcal{N}(\alpha) \equiv 1 \pmod{2}$ . Then the Neumann invariant for the product reduces to that of the 2-torus, i.e.  $\mathcal{N}(\alpha \times \beta) = \mathcal{N}(\alpha)$ . We have considered the Rochlin invariant on these manifolds in Section 4.4.

We now make use of an integer invariant that captures the Rochlin invariant modulo 16, and hence describes the combination of the eta invariants appearing in the global anomaly formula.

**The Fischer-Kreck cobordism invariant.** Following [18], we define the invariant (cf. (4.14))

$$R(Y^{11}, \omega) = -\eta(Y^{11}, S) + 4[h(Y^{11}; D_{TY}) + \eta(Y^{11}; D_{TY})] - 16\eta(Y^{11}; D) \pmod{16} \equiv S(Y^{11}, \omega) . \quad (4.25)$$

The cobordism invariant  $(Y^{11}, \omega) \mapsto S(Y^{11}, \omega) \in \mathbb{R}/\mathbb{Z}$  is an integer described as follows (see [18]). There is an exact sequence

$$0 \rightarrow K \rightarrow \Delta_{10}^{\text{Spin}} \rightarrow \Omega_{10}^{\text{Spin}} \oplus \Omega_{11}^{\text{Spin}} \oplus W_-(\mathbb{Z}; \mathbb{Z}) \rightarrow 0 \quad (4.26)$$

with isomorphism  $K \rightarrow \mathbb{Z}_{16}$  given by  $[X^{10}, \omega, f, h] \mapsto R((X^{10} \times S^1)_f, \omega_h) - \overline{\mathcal{N}}(X^{10}, f) \pmod{16}$ . Here  $\overline{\mathcal{N}} := \mathcal{N} \pmod{16}$ , i.e. is the reduction modulo 16 of the Neumann invariant described above. The sequence in fact splits and, with  $\Omega_{10}^{\text{Spin}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $\Omega_{11}^{\text{Spin}} = 0$ , there is an isomorphism

$$\Delta_{10}^{\text{Spin}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus W_-(\mathbb{Z}; \mathbb{Z}) \oplus \mathbb{Z}_{16} . \quad (4.27)$$

The Witt group  $W_-(\mathbb{Z}; \mathbb{Z})$  is described towards the end of Section 4.2. The new invariant in this case is

$$S(X^{10}, \omega, f, h) : \Delta_{10}^{\text{Spin}} \rightarrow \mathbb{Z}_{16} . \quad (4.28)$$

Note that the exact sequence also shows that every 11-dimensional Spin manifold is cobordant to a mapping torus. Therefore, considering the Fischer-Kreck invariant for mapping tori in fact covers all Spin 11-manifolds, since this invariant is a cobordism invariant.

**Dependence on homotopy and relation to the Ochanine invariant.** Let  $h$  and  $h'$  be representatives of the two homotopy classes of homotopies joining the Spin structure  $\omega$  to Spin structure  $\omega \circ f$ . Then, using [18], we have that the variation of the Fischer-Kreck invariant is given by the Ochanine invariant of the base

$$S(X^{10}, \omega, f, h) - S(X^{10}, \omega, f, h') = O(X^{10}, \omega) . \quad (4.29)$$

This is analogous to the variation of the Rochlin invariant leading to expression (4.20). The appearance of the Ochanine invariant in the dual type IIA string theory has been highlighted in Ref. [43].

**Example 11: Effect of torsion in middle homology.** Consider our standard example, the 10-manifold  $X^{10} = S^5 \times S^5$  and let  $a$  and  $b$  form a basis of  $H_5(X^{10}; \mathbb{Z})$  which is defined by the embedding of the first factor and by the diagonal, respectively. The intersection pairing with respect to this basis is given by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The mod 2 refinement  $q$  given by the normal bundle of an embedded sphere is defined by  $q(x) = 0$  if and only if the normal bundle is trivial. This refinement has values  $q(a) = 0$  and  $q(b) = 1$  on the basis elements. The automorphism of  $H_5(X^{10}; \mathbb{Z})$  defined by  $a \mapsto a + b$ ,  $b \mapsto b$  can be realized by a diffeomorphism  $f$  of  $X^{10}$  which keeps a neighborhood  $U$  of the diagonal  $\Delta(S^5)$  fixed. Then in this case the Rochlin invariant is even  $R(X^{10}, \omega, f, h) \equiv 0 \pmod{2}$  and the Neumann invariant is  $\mathcal{N}(X^{10}, f) = 1$ . Now  $2b$  can be realized by an embedding  $S^5 \hookrightarrow U$  and has a trivial normal bundle, since  $q(2b) = 0$ . This then allows us to do surgery using a tubular neighborhood of  $2b$  contained in  $U$  [18].

To that end, consider the resulting 10-manifold  $\tilde{X}^{10} := (X^{10} \setminus S^5 \times \mathbb{D}^5) \bigcup \mathbb{D}^6 \times S^4$  and diffeomorphism  $\tilde{f} := f|_{X^{10} \setminus S^5 \times \mathbb{D}^5 \cup \text{id}}$ . Since the two manifolds  $(X^{10}, f)$  and  $(\tilde{X}^{10}, \tilde{f})$  are Spin cobordant, the difference of their Rochlin invariants and reduced Neumann invariants (i.e. essentially the Fischer-Kreck invariants) are equal  $(R - \overline{\mathcal{N}})(X^{10}, f) = (R - \overline{\mathcal{N}})(\tilde{X}^{10}, \tilde{f})$ . Furthermore, we have for the homology groups  $H_4(\tilde{X}^{10}) \cong \mathbb{Z}_2 \cong H_5(\tilde{X}^{10})$ , so that the middle cohomology is torsion. Rationally,  $H_5(\tilde{X}^{10}; \mathbb{Q}) = 0$ , which implies that the Neumann invariant vanishes  $\mathcal{N}(\tilde{X}^{10}, \tilde{f}) = 0$  so

that the Rochlin invariant  $R(\tilde{X}^{10}, \tilde{f})$  is a generator of  $\mathbb{Z}_{16}$ . Now for any  $n$ , the Rochlin invariant corresponding to an iterated cobordism satisfies  $R(\tilde{X}^{10}, \tilde{f}^n) = nR(\tilde{X}^{10}, \tilde{f})$ . This means that  $\tilde{X}^{10}$  is a Spin manifold with torsion middle cohomology such that for any integer  $r$  there exists a Spin diffeomorphism  $(f, h)$  on  $(X^{10}, \omega)$  such that  $R(X^{10}, \omega, f, h) \equiv r \pmod{16}$ .

The consequences of the above example are two-fold. First that, as stated towards the end of Section 4.4, that the Rochlin invariant cannot be computed if torsion in middle homology is present. Second, aside from torsion, given a value of the Rochlin invariant corresponding to a diffeomorphism  $f$ , one can produce any integer multiple of this invariant by considering the iterations of  $f$ .

## Acknowledgements

The author would like to thank Fei Han for useful discussions on [21] and for kind hospitality at the University of Singapore in January 2011. The author would like to thank the American Institute of Mathematics, Palo Alto, for hospitality during the program *Algebraic Topology and Physics* in May 2011 as well as IHES, Bures-sur-Yvette, for hospitality in Summer 2011 while this project was being completed. This research is supported by NSF Grant PHY-1102218.

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